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Additive normal forms and integration of differential fractions

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ABSTRACT

This paper presents two new normal forms for fractions of differential polynomials, as well as algorithms for computing them. The first normal form allows to write a fraction as the derivative of a fraction plus a nonintegrable part. The second normal form is an extension of the first one, involving iterated differentiations. The main difficulty in this paper consists in defining normal forms which are linear operations over the field of constants, a property which was missing in our previous works. Our normal forms do not require fractions to be converted into polynomials, a key feature for further problems such as integrating differential fractions, and more generally solving differential equations.

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1. Introduction

This paper defines two new normal forms of differential fractions (i.e. fractions of two differential polynomials) in the context of differential algebra (Ritt, 1950; Kolchin, 1973). The differential polynomial ring $\mathscr{R}$ considered in this paper is as follows: one of its derivation is denoted δ; one assumes that there exists an element x of $\mathscr{R}$ such that $\delta x = 1$; and $\mathscr{K}$ is its field of constants w.r.t. δ (see however Section 2 for the rigorous assumptions on $\mathscr{R}$). The set $\mathscr{S}$ of the so-called differential fractions is defined as the field of fractions of $\mathscr{R}$. A major result of the paper is Proposition 52 which shows that any differential fraction $F \in \mathscr{S}$ can be uniquely decomposed as a sum:

$$F = P + \sum_{i=0}^{\infty} \delta^i W_i\,, \tag{1}$$

where $P \in \mathscr{K}[x]$ is a polynomial, the W_i are differential fractions in the set $\mathscr{S}_{\mathcal{F}} \subset \mathscr{S}$ of the so-called "*functional*" fractions, and where only a finite number of W_i are nonzero. Moreover, we provide Algorithm IteratedIntegrate (see page 37) for computing (1) and prove in Proposition 52 that Normal Form (1) is unique and additive, i.e. that, if

$$\bar{F} = \bar{P} + \sum_{i=0}^{\infty} \delta^i \bar{W}_i$$

is the unique decomposition of some differential fraction $\bar{F} \in \mathscr{S}$ then

$$F + \bar{F} = (P + \bar{P}) + \sum_{i=0}^{\infty} \delta^i (W_i + \bar{W}_i)$$

is the unique decomposition of $F + \bar{F}$. More precisely, in terms of vector spaces, Proposition 52 shows that

$$\mathscr{S} = \mathscr{K}[x] \oplus \mathscr{S}_{\mathcal{F}} \oplus \delta\mathscr{S}_{\mathcal{F}} \oplus \delta^2\mathscr{S}_{\mathcal{F}} \oplus \cdots$$

where $\mathscr{S}$ is seen as a $\mathscr{K}$-vector space.

These results improve those of Boulier et al. (2013) since the decomposition provided by Boulier et al. (2013) depends on the implementation of Boulier et al. (2013, Algorithm integrate) and is not additive. Moreover, (Boulier et al., 2013, Algorithm integrate) is flawed since it may not terminate over some inputs (see Boulier et al., 2014). Our results also extend (Boulier et al., 2014), which fixes the flaw in Boulier et al. (2013, Algorithm integrate) but does not address the additivity property.

Even without the additivity property, algorithms for computing (1) are important: they permit to reduce the size of formulas in the output of differential elimination methods (when polynomials are solved w.r.t. their leading derivatives, the left-hand sides become differential fractions), they give more insight to understand the structure of an equation, and they lead to better numerical schemes in the context of parameter estimation problems over noisy data, from the input–output equations, because they permit to replace, at least partially, numerical derivation methods by numerical integration ones. See Boulier et al. (2014) for details. It is worth mentioning that working with fractions instead of polynomials yields more freedom by adjusting the denominators. Indeed, decomposition (1) highly depends on the denominator of F, i.e. the decomposition of F/Q, where Q is a polynomial, can be completely different from the decomposition of F. Finding a suitable Q is a difficult task and depends on the application (in the context of Boulier et al. (2013), the goal was to obtain order zero W_i).

Variants of Normal Form (1) can be easily obtained, e.g. by bounding the value of i. Bounding i by 1, a unique decomposition of a fraction F can be defined by

$$F = W + \delta R \tag{2}$$

where W is a functional fraction, and R is a fraction. Actually, Normal Form (1) is in practice obtained by iterating Normal Form (2), which is obtained by Algorithm Integrate (see page 33 and