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On a conjecture of Vasconcelos via Sylvester forms

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ABSTRACT

We study the structure of the Rees algebra of an almost complete intersection monomial ideal of finite co-length in a polynomial ring over a field, assuming that the least pure powers of the variables contained in the ideal have the same degree and that the additional monomial has the property that all variables have the same degree. It is shown that the Rees algebra has a natural quasi-homogeneous structure and its presentation ideal is generated by explicit Sylvester forms. A consequence of these results is a proof that the Rees algebra is almost Cohen–Macaulay, thus providing an affirmative partial answer to a conjecture of W. Vasconcelos.

0. Introduction

Let $R := k[x_1, \dots, x_n]$ denote a polynomial ring over a field k. In 2013 W. Vasconcelos formulated the conjecture that the Rees algebra of an Artinian almost complete intersection $I \subset R$ generated by

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monomials is almost Cohen–Macaulay (cf. Hong et al., 2013, Conjecture 4.15). For the binary case (i.e., for $n = 2$) a result of M. Rossi and I. Swanson (Rossi and Swanson, 2003, Proposition 1.9) gives an affirmative answer to the conjecture. Recently, different proofs were established in the binary case of monomials of the same degree as a consequence of work by T.B. Cortadellas and C. D'Andrea (Cortadellas and D'Andrea, 2015), and independently, of work by the present second and third authors (Simis and Tohăneanu, 2015).

Homogeneous ideals $I \subset R$ that are almost complete intersections generated in fixed degree play a critical role in elimination theory of both plane and space parameterizations, while their Rees algebras encapsulate some of the most common tools in both theoretic and applied elimination. Finding a natural set of minimal generators of the presentation ideal of the Rees algebra – informally referred to as *minimal relations* – is a tall order in commutative algebra. It is tantamount to obtaining natural syzygies of the powers of I, a problem that can be suitably translated into elimination theory as the method of moving lines, moving surfaces, and so forth (see e.g., Cox, 2008). One idea to reach for these relations is to draw them in some sort of recursive way out of others already known. One such recurrence is known as the method of Sylvester forms, where certain square *content matrices* are produced. These matrices express the inclusion of two ideals in terms of given sets of generators, where the included ideal is generated by old relations. As an easy consequence of Cramer's rule, the determinant of such a matrix will be a relation. As is well-known, telling whether these relations do not vanish – let alone that they contribute new relations – is a major problem. The determinants of these content matrices, or a construction that generalizes them, are called *Sylvester forms*. The appearance of Sylvester forms goes back at least to the late sixties in a paper of Wiebe (Wiebe, 1969; see also Cox et al., 2008). They have been largely used in many sources, such as (Busé, 2009; Cortadellas and D'Andrea, 2014, 2015; Cox, 2008; Cox et al., 2008; Hassanzadeh and Simis, 2014; Jouanolou, 1997; Hong et al., 2008, 2012, 2013; Simis and Tohăneanu, 2015).

Among the commonly sought features of a Rees algebra is the Cohen–Macaulay property, which is knowingly a certain regularity condition on the ideal. Unfortunately, even binary parameterizations do not typically lead to a Cohen–Macaulay Rees algebra, and since this property has a difficult translation back into elimination theory, why does one care? Well, as it turns, the property ties beautifully with several other properties and current notions of commutative algebra. In this vein, having a little less than the property itself may be useful. This is where *almost Cohen–Macaulay* comes into the picture. Looking for this slightly weaker property is the driving source of some of the references mentioned above and is the main object of this paper as well.

We tackle the case of a monomial parametrization in arbitrary number of variables with an extra condition on the degrees of the monomials, called *uniformity*. Under this condition, we answer affirmatively the stated conjecture. In our opinion this contributes a significant step toward the general case, since we have in mind a couple of procedures to reducing the case of general exponents to this one. Notwithstanding the seemingly simple case of a monomial ideal, as compared to the problem of ideals generated by arbitrary forms – a situation still lacking a bona fide conjecture – its general case may require ahead an additional tour de force beyond the facilitation provided by the methods of the present paper.

Now, one main tool in the binary case of an Artinian almost complete intersections I of forms of the same degree is birationality. The other two tools are the Ratliff–Rush filtration theory and the Huckaba–Marley criterion using a minimal reduction of I. While the Ratliff–Rush filtration gives no insight into the conjectured property of the Rees algebra beyond the two variables case, using the criterion of Huckaba–Marley, would probably require as much calculation and also lead to no reasonable bound to manage the partial lengths. We add the fact that even when the uniformity assumption degenerates into equigrading, birationality for more than two variables is an issue, and hence computing the first Hilbert coefficient of R/I becomes a hardship.

The method in the present paper emphasizes the structure of the presentation ideal of $\mathcal{R}_R(I)$ that may benefit from the appeal to Sylvester forms, as we understand them in their modern algebraic formulation. However, additional muscle work soon became imperative, so the technology comes from three sources: first, a thorough use of the natural quasi-homogeneous grading over k of the presentation ideal of $\mathcal{R}_R(I)$, compatible with the usual standard grading of $\mathcal{R}_R(I)$ over R; second, mastering the overwhelming presence of a sequence of iterated Sylvester forms that are Rees generators; third,