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# Third-order ordinary differential equations equivalent to linear second-order ordinary differential equations via tangent transformations

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ABSTRACT

The linearization problem of a third-order ordinary differential equation by the tangent transformation is considered in the present paper. This is the first application of tangent (essentially) transformations to the linearization problem of third-order ordinary differential equations. Necessary and sufficient conditions for a third-order ordinary differential equation to be linearizable are obtained.



## 1. Introduction

The basic problem in modeling physical phenomena is to find solutions to differential equations. In general, these equations are very difficult to solve explicitly. One of the fundamental methods for solving them makes use of a change of variables that transforms a general solution to a given differential equation into another differential equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, there arises the problem of transforming a given differential equation into a linear equation. This problem is called a *linearization problem*.

In 1997, Grebot (1997) studied the linearization of third-order ordinary differential equations by means of a restricted class of point transformations, namely, $t = \varphi(x)$, $u = \psi(x, y)$, although the prob-

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lem was not completely solved. Complete criteria for linearization by means of point transformations were obtained in Ibragimov and Meleshko (2005). For third-order ordinary differential equations, the linearization problem by a contact transformation has been an interesting topic and was studied in Doubrov (2001), Doubrov et al. (1999), and Gusyatnikova and Yumaguzhin (1999). The solutions to the linearization problem were given in Neut and Petitot (2002) and Ibragimov and Meleshko (2005). The explicit form of the criteria for linearization and the procedure for the construction of the linearizing transformation were presented in Ibragimov and Meleshko (2005).

To find linearization conditions by tangent transformations, one can use a Laguerre form (Laguerre, 1879; Ibragimov, 1996) of a linear ordinary differential equation

$$y^{(n)} + a_{n-3}(x)y^{(n-3)} + \cdots + a_1(x)y' + a_0(x)y = 0.$$

The generalized Sundman transformation takes an intermediate place between point and contact transformations. The linearization problem for a third-order ordinary differential equation was also investigated in Berkovich (1999) and Euler et al. (2003) with respect to a generalized Sundman transformation, i.e.,

$$u(t) = F(x, y), \qquad dt = G(x, y)dx. \tag{1}$$

Criteria for a third-order ordinary differential equation to be equivalent to the linear equation

$$u''' = 0 \tag{2}$$

with respect to the generalized Sundman transformation were presented in Euler et al. (2003). In 2010, Nakpim and Meleshko (2010) obtained necessary and sufficient conditions for a third-order ordinary differential equation to be linearizable into $u''' + \alpha u = 0$ where $\alpha$ is a constant. Some applications of generalized Sundman transformations to ordinary differential equations were considered in Berkovich (2001) and earlier papers, which were summarized in Berkovich (2002). Note that for the linearization problem via generalized Sundman transformations, one needs to use the general form of a linear ordinary differential equation instead of the Laguerre form.

It is known that all aforementioned methods (point, contact, and generalized Sundman transformations) are complementary: there are equations that can be linearized only by one of these methods. A new set of tangent transformations for linearization problems of ordinary differential equations is considered in this paper.

The main tool for solving a linearization problem is a compatibility analysis of overdetermined systems of partial differential equations.[1] In almost all linearization problems, the compatibility analysis consists of comparison of mixed derivatives obtained by different ways. If all derivatives of some order $n$ are found and comparison of all leading cross derivatives does not produce new equations of order at most $n$, then the system is compatible. This study requires a large amount of symbolic calculations involving prolongations of a system, substitution of complicated expressions, and the determination of ranks of matrices. For this purpose, computer symbolic systems can provide valuable help. To obtain results of this paper, we used Reduce system (Hearn, 1987).[2]

Reduce is a symbolic computer package which has capability to deal with several cumbersome calculations, for example, finding derivatives to any specified order and performing arithmetic among complicated functions. It also allows users to make certain assumptions towards any variables in order to ease their computation. The syntax of Reduce is user-friendly: each command is short and analogous to its corresponding mathematical notation. Several packages are available in Reduce, including the package Crack which provides tools for solving overdetermined systems of partial or ordinary differential equations. For more details on Reduce and Crack, see Davenport et al. (1988) and Wolf (2015), respectively.

[1] The modern state in the study of compatibility analysis can be seen in Pommaret (1978), Meleshko (2005), and Gerdt (1999).

[2] A brief review of computer systems of symbolic manipulations can be found, for example, in Davenport (1994).