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Implicitizing rational surfaces using moving quadrics constructed from moving planes

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A B S T R A C T

This paper presents a new algorithm for implicitizing tensor product surfaces of bi-degree (m,n) with no base points, assuming that there are no moving planes of bi-degree $(m-1,n-1)$ following the surface. The algorithm is based on some structural results: (1) There are exactly $2n$ linearly independent moving planes of bi-degree $(m,n-1)$ following the surface; (2) mn linearly independent moving quadrics of bi-degree $(m-1,n-1)$ following the surface can be constructed from the $2n$ linearly independent moving planes; (3) The mn linearly independent moving quadrics form a compact determinant of order mn which exactly gives the implicit equation of the rational surface. Complexity analysis and experimental results show that the new algorithm is significantly more efficient than the previous methods.

1. Introduction

In Computer Aided Geometric Design and Geometric Modeling, the problem of implicitizing rational parametric surfaces has received much attention in the past three decades, since the implicit equation of a parametric surface is an important tool for representing and analyzing the parametric shape.

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Given a rational parametric surface

$$x = \frac{a(s,t)}{d(s,t)}, \quad y = \frac{b(s,t)}{d(s,t)}, \quad z = \frac{c(s,t)}{d(s,t)}$$

where a, b, c, d are polynomials in s, t, there exists a polynomial equation $f(x, y, z) = 0$ which defines the same surface. $f(x, y, z) = 0$ is called the *implicit equation* of the rational surface.

The classical method for finding the implicit equation of a rational parametric surface is to eliminate the variables s and t by computing the bivariate resultant (Dixon, 1908) of the three polynomials $a(s,t) - x \cdot d(s,t)$, $b(s,t) - y \cdot d(s,t)$, $c(s,t) - z \cdot d(s,t)$.

Another traditional implicitizing technique is the Groebner basis method (Buchberger, 1989; Cox et al., 1992). Consider the ideal $I = \langle a(s,t) - x \cdot d(s,t), b(s,t) - y \cdot d(s,t), c(s,t) - z \cdot d(s,t), 1 - d(s,t)w \rangle$ and compute a Groebner basis with respect to a lexicographic ordering, where s, t and w are greater than x, y, z. Let f be the elements of the Groebner basis not involving s, t, w, then $f = 0$ is the implicit equation of the rational parametric surface. Other implicitization techniques can be found in the references in (Sederberg and Chen, 1995).

In 1995, Sederberg and Chen introduced a radical new technique called *moving curves and moving surfaces* to implicitize rational parametric curves and surfaces (Sederberg and Chen, 1995; Sederberg and Saito, 1995; Zhang et al., 1999; Cox et al., 1998a, 1998b, 2000; Chen et al., 2001, 2005, 2008; Chen and Wang, 2003; Busé et al., 2003; Busé and Chardin, 2005; Dohm, 2009; Zheng and Sederberg, 2001). Sederberg and Chen's method of moving quadrics uses only elementary linear algebra—solving a system of linear equations—and as an added bonus represents the implicit equation as the determinant of a matrix one-fourth the size of the classical resultant.

Compared with previous technique, Sederberg and Chen's method has several advantages. First, it can write the implicit equation in a more compact form than other resultant-based methods such as Dixon's resultants (Sederberg et al., 1984; Chionh and Goldman, 1992; Chionh et al., 2000; Chionh and Sederberg, 2001). Second, it always works even in the presence of base points while other resultant-based methods may fail. Third, it is much more efficient than traditional implicitizing techniques such as Groebner basis method (Buchberger, 1989). Unfortunately, there lack explicit constructions and rigorous proofs for the method of moving surfaces for general rational parametric surfaces, except for some special cases (Adkins et al., 2005; Busé et al., 2009; Khetan and D'Andrea, 2006; Shen et al., 2006; Shi and Goldman, 2012; Wang and Chen, 2012).

Cox, Goldman and Zhang (2000) proved that if a rational surface has no base points, then the method of moving quadrics will always work provided that there is no moving plane of low degree that follows the surface. Since the existence of such a moving plane of low degree is represented by a polynomial condition, this result establishes that the method of moving quadrics works for almost all rational surfaces without base points.

In this paper, we will provide a more efficient algorithm to compute the moving quadrics of a parametric surface. Instead of directly solving a linear system of equations, we construct moving quadrics from moving planes. Specifically, suppose that we are given a rational parametric surface of bi-degree (m, n) with no base points and no moving planes of bi-degree $(m-1, n-1)$ following the surface, then we will show that: (1) there are exactly $2n$ linearly independent moving planes of bi-degree $(m, n-1)$ following the surface; (2) we can construct mn linearly independent moving quadrics of bi-degree $(m-1, n-1)$ from the $2n$ linearly independent moving planes of bi-degree $(m, n-1)$; (3) the mn moving quadrics can be used to construct a compact determinant of order mn which gives the implicit equation of the rational surface.

The paper is structured as follows. Section 2 recalls some preliminary knowledge about moving surface method for surface implicitization. Section 3 describes the framework of our algorithm for implicitizing rational surfaces with moving quadrics. Section 4 presents theoretic results to support the algorithm. Section 5 provides the details of the main algorithm. Complexity analysis of the algorithm and experimental results are provided in Section 6 together with a comparison with direct method of moving quadrics. Finally in Section 7, we conclude the paper with some open problems for future research.