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Estimating the number of tetrahedra determined by volume, circumradius and four face areas using Groebner basis ☆

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A R T I C L E I N F O

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Given any set of six positive parameters, the number of tetrahedra, all having these values as their volume, circumradius and four face areas, is studied. We identify all parameters that determine infinitely many tetrahedra. On the other hand, we classify parameters that determine finitely many tetrahedra and find only four different upper bounds, zero, six, eight, and nine, on the numbers of tetrahedra. In each case, the upper bound is sharp in the complex domain.

In this paper, the upper bounds are obtained through checking the dimensions of various quotient algebras of ideals by counting monomials. This is done by computing Groebner bases with block orders. Partitioning the parameter space into several cases, we find either the dimension or an upper bound of it for the quotient algebra in each case. From that, various upper bounds on the number of tetrahedra are obtained. To show the upper bounds are sharp, we pick rational parameters and study the number of tetrahedra through Hermite's root counting method.

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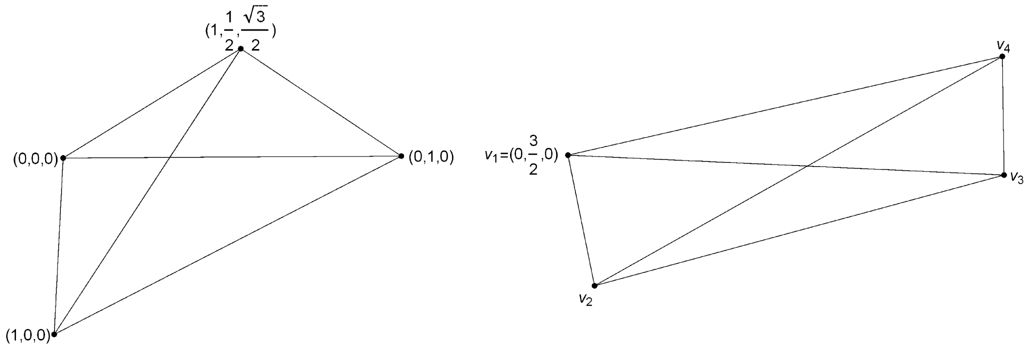

Fig. 1. Two tetrahedra.

1. Introduction

In 1999, M. Mazur defined a tetrahedron to be *rigid* if there are no other tetrahedra with the same volume, radius of circumscribed sphere, and areas of faces (Mazur, 1999). He proposed a question asking whether every tetrahedron is *rigid*. Specifically, for given six positive constants V, R, A_1, A_2, A_3, A_4, when there is a tetrahedron having these values as its volume, circumradius and four face areas, respectively, is such tetrahedron unique?

This question was answered negatively by Lisoněk and Israel in 2000 (Lisoněk and Israel, 2000). They found that, when $V=\sqrt{\frac{1}{48}}$, $R=\sqrt{\frac{7}{12}}$, $A_1=A_2=\sqrt{\frac{7}{16}}$, $A_3=A_4=\frac{1}{2}$, there are two different tetrahedra having these values as their volume, circumradius and four face areas, respectively. They are shown in Fig. 1, where the tetrahedron on the left side has squares of edge lengths 1, 2, 1, 1, 2, 2 and the other tetrahedron has squares of edge lengths α_1, β_1, β_2, β_2, β_1, γ_1 with $\alpha_1\approx 0.59$, $\beta_1\approx 1.71$, $\beta_1\approx 2.09$, $\gamma_1\approx 0.69$ and vertices $v_1=(0,1.5,0)$, $v_2\approx(0.77,1.5,0)$, $v_3\approx(0.14,2.8,0)$, $v_4\approx(0.63,2.6,0.65)$.

In the same paper by Lisoněk and Israel, they posed more questions. They ask whether for any positive constants V, R, A_1, A_2, A_3, A_4, there are *finitely many* tetrahedra, all having these values as their respective metric invariants. And, if there are finitely many tetrahedra, what are the upper bounds?

In 2005, Yang and Zeng presented a negative solution in Yang and Zeng (2005) and Mucherino et al. (2013). When $V=441$, $R=\frac{43\sqrt{3}}{6}$, $A_1=84\sqrt{3}$, $A_2=A_3=A_4=63\sqrt{3}$, they found a family of tetrahedra $T_{(x,y)}$, where (x,y) varies over a component of a cubic curve such that all tetrahedra $T_{(x,y)}$ share the same metric invariants. The example they proposed is in the case of $A_2=A_3=A_4$. They made a conjecture that, when A_1, A_2, A_3, A_4 are pairwise distinct, such metric invariants indeed determine finitely many tetrahedra. They also conjectured an upper bound nine in these cases.

Yang and Zeng, in 2013, proved their conjectures by claiming that, given six positive numbers V, R, A_1, A_2, A_3, A_4, there are at most eight tetrahedra with volume V, circumradius R and four face areas A_1, A_2, A_3, A_4, except in the case that three of the values A_1, A_2, A_3, A_4 are equal (Yang and Zeng, 2013). The upper bound eight is obtained in the real domain. The cases of three equal face areas are not discussed there.

In McConnell (2012), McConnell introduced the concept of *pseudo-faces* and used a different system from that in Lisoněk and Israel (2000) and Yang and Zeng (2005) to study the same problem. There, the example of Yang and Zeng in 2005 is generalized. Also, finiteness results are proved in the cases of $A_1=A_2=A_3=A_4$ and $A_1=A_2\neq A_3=A_4$. However, the cases with only two equal face areas are not discussed. Also, none of the upper bounds are given in the paper McConnell (2012).

In this paper, we use a parametric polynomial system with three *pseudo-faces* introduced by McConnell as variables and V, R, A_1, A_2, A_3, A_4 as parameters to study the numbers of tetrahedra. The method used here is different from those in Yang and Zeng (2013) and McConnell (2012). We mainly apply Groebner basis in studying our parametric polynomial system. Both the finiteness