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Propagating weights of tori along free resolutions

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ABSTRACT

The action of a torus on a graded module over a polynomial ring extends to the entire minimal free resolution of the module. We explain how to determine the action of the torus on the free modules in the resolution, when the resolution can be calculated explicitly. The problem is reduced to analyzing how the weights of a torus propagate along an equivariant map of free modules. The results obtained are used to design algorithms which have been implemented in the software system Macaulay2.

1. Introduction

This paper is structured as follows. In this section you will find an overview of the questions we address (Section 1.1) followed by an example (Section 1.2) presented with the minimum amount of technical background. Section 2 introduces some basic concepts of commutative algebra and the representation theory of tori, and proceeds to describe their natural interactions. In Section 3, we analyze how weights of tori propagate along equivariant maps of free modules, first in the easier case of bases of weight vectors (Section 3.1) and then in a more general setting (Section 3.2). Our last section is devoted to the design of various algorithms: to propagate weights along an equivariant map of free modules from codomain to domain (Section 4.1), to propagate weights 'forward' from domain to codomain (Section 4.2), for resolutions (Section 4.3), and, as a bonus, an algorithm to determine

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the weights of graded components of modules (Section 4.4). Finally, in Section 4.5, we discuss the possibility of carrying out all such computations over subfields.

An implementation of the algorithms of this paper for semisimple complex algebraic groups is included, under the package name `HighestWeights`, with version 1.7 of the software system `Macaulay2` (Grayson and Stillman, 2015) and is documented in Galetto (2015).

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1.1. *Motivation*

Every finitely generated module over a polynomial ring with coefficients in a field has a finite minimal free resolution which is unique up to isomorphism. It is typically used to produce numerical invariants such as projective dimension, regularity, (graded) Betti numbers and the Hilbert series of a module. While there are descriptions for certain classes of modules, finding a minimal free resolution of a module is, in general, a very difficult problem. Computational methods offer a solution to this problem in many cases, although they are limited in scope by time and memory constraints. As the matrices of the differentials grow in size, their description is often omitted.

Consider the case of a polynomial ring A endowed with an action of a group G which is compatible with grading and multiplication (see Section 2.3 for the precise definitions). Let us denote $\mathrm{mod}_{\circlearrowleft G} A$ the category of finitely generated graded A-modules with a compatible action of G and homogeneous G-equivariant maps. If M is an object in $\mathrm{mod}_{\circlearrowleft G} A$, then the action of G extends to the entire minimal free resolution of M. A free A-module F is isomorphic to $(F/\mathfrak{m}F) \otimes A$, where $\mathfrak{m}$ denotes the maximal ideal generated by the variables of A. The representation theoretic structure of F, i.e. the action of G on F, is then controlled by the representation $F/\mathfrak{m}F$. Therefore, if the complex $F_\bullet$:

$$0 \to F_n \xrightarrow{d_n} F_{n-1} \to \ldots \to F_i \xrightarrow{d_i} F_{i-1} \to \ldots \to F_1 \xrightarrow{d_1} F_0$$

denotes a minimal free resolution of M, we could try to determine the action of G on each representation $F_i/\mathfrak{m}F_i$.

The representation theoretic structure of $F_\bullet$ may offer some insight into the maps of the complex. Consider the situation of a differential $d_i \colon F_i \to F_{i-1}$, with $F_i/\mathfrak{m}F_i$ an irreducible representation of G. The map d_i is completely determined by its image on a basis of F_i; hence we can reduce to a map of representations $F_i/\mathfrak{m}F_i \to F_{i-1}/\mathfrak{m}F_{i-1} \otimes A$. If the decomposition of the tensor product in the codomain contains only one copy of the irreducible $F_i/\mathfrak{m}F_i$ in the right degree, then Schur's lemma (Lang, 2002, Ch. XVII, Prop. 1.1) implies that the map is uniquely determined up to multiplication by a constant. In some cases, this information is enough to reconstruct the map completely (see Galetto, 2013 for a few examples).

The representation theoretic structure of $F_\bullet$ may also be used to determine the class $[M]$, of a module M, in $K_0^G(\mathrm{mod}_{\circlearrowleft G} A)$, the equivariant Grothendieck group of the category $\mathrm{mod}_{\circlearrowleft G} A$. By construction of the Grothendieck group,

$$[M] = \sum_{i=0}^{n} (-1)^i [F_i],$$

where the right hand side is the equivariant Euler characteristic of the complex $F_\bullet$.

Motivated by the discussion above, we pose the following question: is it possible to determine the action of a group on a minimal free resolution of a module computationally? The first assumption is that the resolution itself can be computed explicitly in a reasonable amount of time. Secondly, we restrict to a class of groups whose representation theory is well understood and manageable: tori. Every representation of a torus is semisimple, with irreducible representations being one dimensional and parametrized by weights. Moreover, weights can be conveniently represented by integer vectors. More importantly, finite dimensional representations of connected reductive algebraic groups over