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Knapsack problems in products of groups ☆

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ABSTRACT

The classic knapsack and related problems have natural generalizations to arbitrary (non-commutative) groups, collectively called knapsack-type problems in groups. We study the effect of free and direct products on their time complexity. We show that free products in certain sense preserve time complexity of knapsack-type problems, while direct products may amplify it. Our methods allow to obtain complexity results for rational subset membership problem in amalgamated free products over finite subgroups.

1. Introduction

In Myasnikov et al. (2015), the authors introduce a number of certain decision, search and optimization algorithmic problems in groups, such as the subset sum problem, the knapsack problem, and the bounded submonoid membership problem (see Section 1.1 for definitions). These problems are collectively referred to as *knapsack-type* problems and deal with different generalizations of the

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classic knapsack and subset sum problems over $\mathbb{Z}$ to the case of arbitrary groups. In the same work, the authors study time complexity of such problems for various classes of groups, for example for nilpotent, metabelian, hyperbolic groups. With that collection of results in mind, it is natural to ask what is the effect of group constructions on the complexity of knapsack-type problems, primarily the subset sum problem. In the present paper we address this question in its basic variation, for the case of free and direct products of groups.

Solutions to many algorithmic problems carry over from groups to their free products without much difficulty. It certainly is the case with classic decision problems in groups such as the word, conjugacy (Lyndon and Schupp, 2001, for instance) and membership (Mikhailova, 1968) problems. In some sense, the same expectations are satisfied with knapsack-type problems, albeit not in an entirely straightforward fashion. It turns out that knapsack-type problems such as the aforementioned subset sum problem, the bounded knapsack problem, and the bounded submonoid membership problem share a certain common ground that allows to approach these problems in a unified fashion, and to carry solutions of these problems over to free products. Thus, our research both presents certain known facts about these algorithmic problems in a new light, and widens the class of groups with known complexity of the knapsack-type problems. Our methods apply more generally, which allows us to establish in Section 4 complexity results for certain decision problems, including the rational subset membership problem, in free products of groups with finite amalgamated subgroups.

Algorithmic problems in a direct product of groups can be dramatically more complex than in either factor, as is the case with the membership problem, first shown in Mikhailova (1958). By contrast, the word and conjugacy problems in direct products easily reduce to those in the factors. In Section 3 we show that direct product does not preserve polynomial time subset sum problem (unless $\mathbf{P} = \mathbf{NP}$). Thus, the subset sum problem occupies an interesting position, exhibiting features of both word problem and membership problem; on the one hand, its decidability clearly carries immediately from factors to the direct product, while, on the other hand, its time complexity can increase dramatically.

Below we provide basic definitions and some of the immediate properties of the problems mentioned above. We refer to Myasnikov et al. (2015, 2014) for the initial motivation for the study of non-commutative discrete optimization, the set-up of the problems, and initial facts on non-commutative discrete optimization.

1.1. Preliminaries

In this paper we follow terminology and notation introduced in Myasnikov et al. (2015). For convenience, below we formulate the algorithmic problems mentioned in Section 1. We collectively refer to these problems as *knapsack-type* problems in groups.

Elements in a group G generated by a finite or countable set X are given as words over the alphabet $X \cup X^{-1}$. As we explain in the end of this section, the choice of a finite X does not affect complexity of the problems we formulate below. Therefore, we omit the generating set from notation.

Consider the following decision problem. Given $g_1, \ldots, g_k, g \in G$, and m that is either a unary positive integer or the symbol ∞, decide if

$$g = g_1^{\varepsilon_1} \ldots g_k^{\varepsilon_k} \tag{1}$$

for some integers $\varepsilon_1, \ldots, \varepsilon_k$ such that $0 \le \varepsilon_j \le m$ for all $j = 1, 2, \ldots, k$. Depending on m, special cases of this problem are called:

($m = \infty$) **The knapsack problem KP**(G). We omit ∞ from notation, so the input of **KP**(G) is a tuple $g_1, \ldots, g_k, g \in G$.

($m < \infty$) **The bounded knapsack problem BKP**(G). The input of **BKP**(G) is a tuple $g_1, \ldots, g_k, g \in G$, and a number 1^m.

($m = 1$) **The subset sum problem SSP**(G). We omit $m = 1$ from notation, so the input of **SSP**(G) is a tuple $g_1, \ldots, g_k, g \in G$.