Journal of Symbolic Computation 74 (2016) 125–139

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Integrability test for evolutionary lattice equations of higher order

V.E. Adler

L.D. Landau Institute for Theoretical Physics, Ak. Semenov str. 1-A, 142432 Chernogolovka, Russian Federation

ARTICLE INFO

Article history:
Received 9 February 2015
Accepted 5 May 2015
Available online 14 June 2015

Keywords:
Volterra type lattice
Higher symmetry
Conservation law
Integrability test
Summation by parts
Computer algebra

ABSTRACT

A generalized summation by parts algorithm is presented for solving of difference equations of the form $T^m(y) - ay = b$ with variable coefficients, where T denotes the shift operator. Solvability of equations of this type with respect to the coefficients of formal symmetry (or formal recursion operator) provides a convenient integrability test for evolutionary differential-difference equations $u_{,t} = f(u_{-m}, \dots, u_m)$.

1. Introduction

Let $\mathcal{F}$ be the set of locally analytic functions depending on a finite number of variables $u_j \in \mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $j \in \mathbb{Z}$, and let the shift operator T act on elements of $\mathcal{F}$ according to the rule

$$T(f(u_i, \dots, u_j)) = f(u_{i+1}, \dots, u_{j+1}).$$

Our goal in this article is to develop an algorithm which allows one either to construct a solution $y \in \mathcal{F}$ of the linear difference equation

$$T^m(y) - ay = b, \tag{1}$$

with an integer exponent $m \geq 1$ and given coefficients $a, b \in \mathcal{F}$, or to prove that the solution does not exist. In the latter case, the algorithm should return an *obstacle*, that is, some non-zero expression in terms of the coefficients which would vanish if there was a solution. An important special case

E-mail address: adler@itp.ac.ru.

http://dx.doi.org/10.1016/j.jsc.2015.05.008
0747-7171/

$$(T-1)(y)=b \tag{2}$$

can be reformulated as the inversion problem for the operator of total difference $T-1$. It was addressed by many authors in the context of discrete calculus of variations (Kupershmidt, 1985; Hydon and Mansfield, 2004; Mansfield and Quispel, 2005), see also Olver (1993) for a parallel continuous theory. In particular, it is well known that $\mathbb{K}\oplus\mathrm{im}(T-1)=\ker\mathsf{E}$, where $\mathsf{E}=\sum T^{-j}\partial_j$ is the difference Euler operator or the variational derivative. The preimage of $T-1$ can be computed by use of the so-called summation by parts algorithm or by use of a discrete homotopy operator (Hereman et al., 2006). A variational interpretation of more general operators T^m-a and generalization of discrete homotopy operator remain open questions, very interesting from theoretical standpoint; however, this approach can hardly give an effective solution method for equation (1).

A motivation for study of equation (1) comes from the theory of differential-difference (or lattice) evolutionary equations of the form

$$\partial_t(u_n)=f(u_{n-m},\dots,u_{n+m}),\quad n\in\mathbb{Z}. \tag{3}$$

Recall, that such an equation is called integrable if it is consistent with an infinite set of evolutionary flows of higher orders, or symmetries. Although this property may be not so easy to check intermediately, one can deduce that it implies solvability of certain sequence of equations of the form (1)

$$T^m(g_j)-a_jg_j=b_j,\quad j=0,-1,\dots, \tag{4}$$

where a_j and b_j are computed explicitly if $g_0,\dots,g_{j+1}$ are already known (see Proposition 17). This gives us a test which amounts to stepwise checking of whether equation (4) is solvable with respect to g_j: if not, then equation (3) is not integrable, if so, then we have to compute g_j and to go to the next condition for g_{j-1}. In practice, this test turns out to be very useful, although checking of infinite number of conditions is formally needed in order to prove the integrability. Moreover, the sequence of necessary integrability conditions encoded in relations (4) can be used for classification of the whole set of equations under consideration (this problem is certainly much more difficult than testing of a given equation and may require additional assumptions such as the existence of an infinite set of higher order conservation laws). In the continuous setup, this approach made it possible to solve a number of classification problems for integrable partial differential equations of Korteweg–de Vries and nonlinear Schrödinger type, see e.g. Sokolov and Shabat (1984), Mikhailov et al. (1987), Mikhailov et al. (1991), Mikhailov and Shabat (1993), Meshkov and Sokolov (2012). An exhaustive classification of integrable equations (3) at $m=1$ (Volterra type lattices) was obtained by Yamilov (1983), but only few examples are known at $m>1$ so far, the Bogoyavlensky lattices (Bogoyavlensky, 1991) being the most well studied ones. Several classification results for other types of lattice equations were obtained by Shabat and Yamilov (1991), Adler and Shabat (1997), Adler et al. (2000), Yamilov (2006), Adler (2008). The problems of symbolic computation of the higher symmetries, conservation laws, recursion operators and Lax pairs were discussed in many papers, see e.g. Göktaş and Hereman (1999), Hickman and Hereman (2003), Hereman et al. (2005), Sokolov and Wolf (2001), Tsuchida and Wolf (2005); integrability tests based on these notions were developed e.g. in Gerdt et al. (1985), Gerdt (1993), Hereman et al. (1998).

In the problem of solving equation (1), the main issue is related with the complexity of the coefficients a, b. In particular, the right hand side in (4) becomes more and more involved at each step, so that applying the test may turn to be non-trivial. A simple method described in Section 2.2 makes use of the expansion $(T^m-a)^{-1}=T^{-m}(1+aT^{-m}+(aT^{-m})^2+\dots)$ and allows us to obtain the general solution of (1) in explicit form, if it exists. This approach is rather straightforward, but, unfortunately, it is practically applicable only if the coefficients are not too complicated.

Section 2.3 contains a more effective 'generalized summation by parts algorithm', the main result of the article. In short, it is based on simplifications of equation by a sequence of suitable substitutions which exist under certain relations between a and b. If all relations are fulfilled then we construct the solution after a finite number of steps and if some relation fails then this proves that the solution does not exist.

Applications to the integrability problem for the lattice equations are given in Section 3.1. It contains some basic notions and theorems of the symmetry approach which are necessary in order to