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Nearly optimal refinement of real roots of a univariate polynomial

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ABSTRACT

We assume that a real square-free polynomial A has a degree d, a maximum coefficient bitsize τ and a real root lying in an isolating interval and having no nonreal roots nearby (we quantify this assumption). Then we combine the *Double Exponential Sieve* algorithm (also called the *Bisection of the Exponents*), the bisection, and Newton iteration to decrease the width of this inclusion interval by a factor of $t = 2^{-L}$. The algorithm has Boolean complexity $\widetilde{\mathcal{O}}_B(d^2\tau + dL)$. This substantially decreases the known bound $\widetilde{\mathcal{O}}_B(d^3 + d^2L)$ and is optimal up to a polylogarithmic factor. Furthermore we readily extend our algorithm to support the same upper bound on the complexity of the refinement of r real roots, for any $r \leq d$, by incorporating the known efficient algorithms for multipoint polynomial evaluation. The main ingredient for the latter is an efficient algorithm for (approximate) polynomial division; we present a variation based on structured matrix computation with quasi-optimal Boolean complexity.

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1. Introduction

The problem of the approximation of the real roots of a univariate polynomial appears very often as an important ingredient of various algorithms in computer algebra and nonlinear computational geometry, for example the algorithms that compute the topology of real plane algebraic curves (Cheng et al., 2009; Kerber and Sagraloff, 2011), solve the systems of polynomial equations (Mantzaflaris et al., 2011; Diochnos et al., 2009; Emeliyanenko and Sagraloff, 2012), isolate the real roots of polynomials with coefficients in an extension field (Strzeboński and Tsigaridas, 2011; Johnson, 1991), or compute cylindrical algebraic decomposition (Basu et al., 2006; Davenport, 1988). Typically one starts the approximation of a real root with its isolation, by including it into an interval that contains no other roots. This gives a crude initial approximation, and then one rapidly refines this approximation, e.g., by Newton's method. In the present paper we study just the refinement stage, that is, given a polynomial A, which has a degree d and a maximum coefficient bitsize τ, and an interval with rational endpoints that contains one of its real roots (an isolating interval), we devise an algorithm that refines this *inclusion interval* to decrease its width by a factor $t = 2^{-L}$, for some positive integer L.

The overall Boolean complexity of the known real root-finding algorithms is the same as for complex root-finders, obtained in Pan (1995) (see also Pan, 1996, 2002; McNamee and Pan, 2013, Chapter 15), that is, $\widetilde{\mathcal{O}}_B(d^3 + d^2 L)$. For the complex root refinement, this is within polylogarithmic factors from the optimum provided $\tau = \mathcal{O}(L)$. The upper bound is also the record bound for the real root-refinement problem, but the lower bound does not apply to the real case anymore, and our new record upper bound $\widetilde{\mathcal{O}}_B(d^2\tau + dL)$ of this paper covers the complexity of the refinement of a single real root as well as all real roots of a univariate polynomial. Even for the refinement of an approximation to a single root, this bound is nearly optimal, being within polylogarithmic factor from the information lower bound. Next we supply more details and recall some related results.

The known real root-finders and root-refiners are most efficient in the important special case where the polynomial has only real roots. In this context we refer to the work of Ben-Or and Tiwari (1990) that introduced interlacing polynomials and *Double Exponential Sieve*. Pan and Linzer (1999) and Bini and Pan (1998), see also Bini and Pan (1991, 1992), modified the approach of Ben-Or and Tiwari (1990) (they called it *Bisection of Exponents*) to approximate the eigenvalues of a real symmetric tridiagonal matrix by using Courant–Fischer minimax characterization theorem. In Pan et al. (2007) a variant of the refinement algorithms in Pan and Linzer (1999), Bini and Pan (1998) is used, for the approximation of all the real roots of a polynomial. We also wish to cite the real root-finders (for polynomials with only real roots) by means of Quasi-Laguerre Iteration (Du et al., 1996, 1997). They have highly efficient implementation and, like the algorithms cited above, also support nearly optimal complexity bounds for this task.

Collins and Krandick (1993) presented a variant of Newton's algorithm where all the evaluations involve only dyadic numbers, as well as a comparison with the case where operations are performed with rationals of arbitrary sizes. Quadratic convergence of Newton's iterations is guaranteed by point estimates and α-theory of Smale, e.g. Blum et al. (1998), Renegar (1987). For robust approximation of zeros based on bigfloats operations we refer the reader to Sharma et al. (2005). A very interesting and efficient algorithm that combines bisection and Newton iterations is the *Quadratic Interval Refinement* (QIR) by Abbott (2014). For a detailed analysis of the Boolean complexity of QIR we refer the reader to Kerber (2009). Kerber and Sagraloff (2011) modify QIR to use interval arithmetic and approximations; they achieve a bound of $\widetilde{\mathcal{O}}_B(d^3\tau^2 + dL)$. A factor of τ could be saved if we use fast algorithms for root isolation of univariate polynomials, e.g. Sagraloff (2012), Pan (2002), Schönhage (1982b). Such an approach is used in Mehlhorn et al. (2015) that makes adaptive the algorithm of Pan (2002) for root approximation and achieves a bound of $\widetilde{\mathcal{O}}_B(d^3 + d^2\tau + dL)$ for refinement. The algorithm of Rouillier (2010), based on Kantorovich point estimates, is efficient in practice but has unknown complexity.

We revisit the approach of Ben-Or and Tiwari (1990), Pan and Linzer (1999), Bini and Pan (1998) to devise our *Real Root Refinement* ($\mathtt{R}_3$) algorithm and present a detailed analysis under the bit complexity model, based on exact operations with rationals (Theorem 12). We also introduce an approximate variant ($\alpha\mathtt{R}_3$) based on interval arithmetic, Section 2.1, where we use multi-precision floating point numbers for computations and for the representation of the endpoints of intervals, and where we