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Lifting Markov bases and higher codimension toric fiber products

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ABSTRACT

We study how to lift Markov bases and Gröbner bases along linear maps of lattices. We give a lifting algorithm that allows to compute such bases iteratively provided a certain associated semigroup is normal. Our main application is the toric fiber product of toric ideals, where lifting gives Markov bases of the factor ideals that satisfy the compatible projection property. We illustrate the technique by computing Markov bases of various infinite families of hierarchical models. The methodology also implies new finiteness results for iterated toric fiber products.

1. Introduction

Let $\mathcal{B} \in \mathbb{Z}^{h\times n}$ be an integer matrix and let $\mathcal{M} \subseteq \mathbb{Z}^n$. For any $b \in \mathbb{Z}^h$ let $\mathbf{F}(\mathcal{B}, b)_{\mathcal{M}}$ be the *fiber graph* with vertex set $\mathbf{F}(\mathcal{B}, b) = \{v \in \mathbb{N}^n : \mathcal{B}v = b\}$, where vertices $u, v \in \mathbf{F}(\mathcal{B}, b)$ are connected by an edge if and only if $u - v \in \pm\mathcal{M}$. Then $\mathcal{M}$ is called a *Markov basis* if all fiber graphs are connected. Elements of Markov bases are sometimes called *moves*, since they can be used as moves in MCMC simulations to

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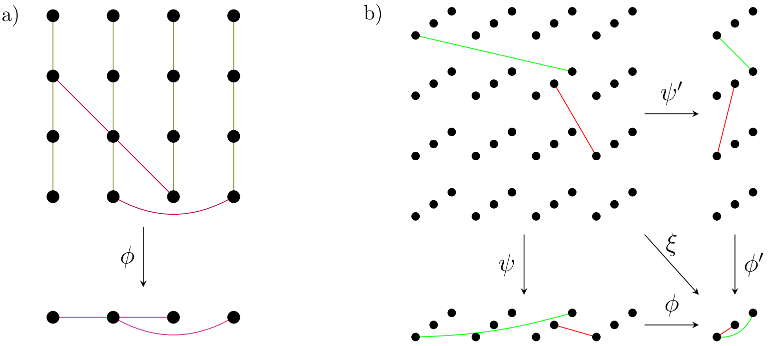

Fig. 1. a) Consider a graph G with vertex set $V \subseteq \mathbb{Z}^n$ and a linear map $\phi : \mathbb{Z}^n \to \mathbb{Z}^t$. If the image of G is connected and if each ϕ-fiber of G is connected, then G itself is connected. The edges in the lower graph correspond to a PF Markov basis. The vertical edges in the upper graph correspond to a kernel Markov basis. The remaining edges are lifts of the PF Markov basis. b) An illustration of the algorithm applied to the toric fiber product: The goal is to lift along the map ξ. This can be accomplished in two steps, by first lifting along ϕ and ϕ' and by then gluing the results.

sample from $\mathbf{F}(\mathcal{B}, b)$ (Diaconis and Sturmfels, 1998). Alternatively, Markov bases consist of exponent vectors of a binomial generating set of the toric ideal $I_{\mathcal{B}}$ (see Theorem 4).

The best general algorithm to compute a Markov basis of a matrix is the one implemented in `4ti2` (4ti2 team, 2003). However, many matrices that appear in applications are too large, and `4ti2` cannot compute a Markov basis within a reasonable time, using a reasonable amount of memory. In these situations, one hopes for procedures that take into account the structure of the Markov basis problem and that can use that structure to build a Markov basis of a large problem from Markov bases of simpler pieces and “lifting” operations.

In this paper we study how to lift a Markov basis along a linear map. The lifting procedure generalizes similar prior constructions. For example, the algorithm implemented in `4ti2` relies on lifting Markov bases along a coordinate projection (Hemmecke and Malkin, 2009). The construction used to compute a Markov basis of codimension zero toric fiber products is also an instance of lifting (Sullivant, 2007). Similar ideas are used in Shibuta (2012) to relate an ideal with its preimage under a monomial ring homomorphism. We study lifting in a very general context for arbitrary matrices $\mathcal{B}$ and arbitrary linear maps ϕ. The only assumption that we have to make is that a certain affine semigroup is normal (see Section 3.1). Even if this condition is violated, in many cases it is possible to adjust our algorithm. An example is given in Section 5.2.

Our procedure allows to transform the problem of computing a Markov basis of $\mathcal{B}$ into a series of smaller Markov basis computations. The efficiency of lifting crucially depends on the choice of the linear map. If everything goes well, it is possible to compute complicated Markov bases of large matrices inductively by iterating the lifting procedure.

The idea behind lifting is sketched in Fig. 1a): For a linear map $\phi : \mathbb{Z}^n \to \mathbb{Z}^t$ and a graph $G = (V, E)$ with $V \subseteq \mathbb{Z}^n$ define the image graph $\phi(G) = (V', E')$ by $V' = \phi(V)$ and $(x', y') \in E'$ if and only if there is $(x, y) \in E$ with $x' = \phi(x)$ and $y' = \phi(y)$. If G is a fiber graph of $\mathcal{B}$ with respect to a Markov basis, then G is connected, and so is $\phi(G)$. Our approach is to turn this observation around as follows: Given a graph homomorphism ϕ induced by a linear map as above, if its image $\phi(G)$ is connected and if each fiber $G[\phi^{-1}(x)] := (\phi^{-1}(x), \{(u, v) \in E : u, v \in \phi^{-1}(x)\}$ is connected, then G is connected. Thus, our strategy is as follows: First, we find a set of moves that connects the ϕ-fibers (that is the sets of the form $G[\phi^{-1}(x)]$). Such a set of moves we call a *kernel Markov basis*, because $\phi^{-1}(x) = V \cap (u + \ker_{\mathbb{Z}} \phi)$ for any $u \in \phi^{-1}(x)$. Second, we find a set of moves that connects the projected fiber graphs $\phi(G)$. Such a set of moves we call a *projected fiber (PF) Markov basis*. Then we lift the PF Markov basis to obtain suitable moves in $\mathbb{Z}^n$. In the last step, the actual lifting step, we need to find “enough” preimages of the edges of the image graph. In Section 3 we give a general lifting algorithm of which the central step is again a Markov basis computation.