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Recognizing implicitly given rational canal surfaces

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ABSTRACT

It is still a challenging task of today to recognize the type of a given algebraic surface which is described only by its implicit representation. In this paper we will investigate in more detail the case of canal surfaces that are often used in geometric modelling, Computer-Aided Design and technical practice (e.g. as blending surfaces smoothly joining two parts with circular ends). It is known that if the squared medial axis transform is a rational curve then so is also the corresponding surface. However, starting from a polynomial it is not known how to decide if the corresponding algebraic surface is a rational canal surface or not. Our goal is to formulate a simple and efficient algorithm whose input is a polynomial with the coefficients from some subfield of $\mathbb{R}$ and the output is the answer whether the surface is a rational canal surface. In the affirmative case we also compute a rational parameterization of the squared medial axis transform which can be then used for finding a rational parameterization of the corresponding implicitly given canal surface.

1. Introduction and related work

In this paper we will pay an attention to the so called canal surfaces. These surfaces, which are defined as envelopes of moving spheres in 3-space, are very popular in Computer-Aided Design as they

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are often used as blending surfaces between the parts with circular ends. It was proved in Peternell and Pottmann (1997) that any canal surface with a rational spine curve (a set of all centers of moving spheres) and a rational radius function possesses a rational parameterization. An algorithm for generating rational parameterizations of canal surfaces was developed and investigated in Landsmann et al. (2001). The class of rational canal surfaces with a rational spine curve and a rational radius function is a proper subset of the class of rational canal surfaces – all canal surfaces with a rational spine curve and rational squared radius function admit a rational parameterization, cf. Peternell (1998), Bastl et al. (2014). The reverse problem (a rational parametric description of a curve or a surface is given, find the corresponding implicit equation) is called the implicitization problem. An algorithm for computing the implicit equation of a canal surface generated by a rational family of spheres is presented in Dohm and Zube (2009).

However our goal is different – we start with an implicit representation (i.e., with some polynomial in x, y, z) and want to decide if the corresponding algebraic surface is a rational canal surface or not. This is still a challenging problem, only partially solved in the recent past for a subfamily of canal surfaces, namely for the surfaces of revolution – see Vršek and Lávička (2015). Moreover, in case of the positive answer we want to compute the equation of the spine curve and also the radius function. We would like to emphasize that this study is interesting not only from the theoretical point of view but it also reflects a need of the real-world applications as the results of many geometric operations are often described only implicitly. Then it is a challenging task to recognize the type of the obtained surface, find its characteristics and for the rational surfaces compute also their parameterizations. This is needed e.g. when the implicit blend surfaces (often of the canal-surface type) are constructed, see Hoffmann and Hopcroft (1985), Rockwood (1989), Hartman (1990), (2001).

Now, we start with short recalling some elementary notions. Let $\mathbb{E}^3_{\mathbb{R}}$ be the Euclidean 3-space equipped with the Cartesian coordinates. A point $\mathbf{x}$ is represented with respect to the coordinate system by the vector (x, y, z), and we do not distinguish between the point and its coordinate vector. Points in the projective closure of $\mathbb{E}^3_{\mathbb{R}}$ will be described using standard homogeneous coordinates

$$(W : X : Y : Z) = (1 : x : y : z). \tag{1}$$

The equation $W = 0$ describes the *ideal plane* as the set of all asymptotic directions, i.e., of points at infinity. The subset of the ideal plane which is invariant with respect to all similarities is called the *absolute conic section* Ω and characterized by

$$\Omega : \; X^2 + Y^2 + Z^2 = W = 0, \tag{2}$$

consisting solely of imaginary points. The absolute conic section allows to characterize circles as conic sections which intersect Ω in two distinct points. Moreover, circles lying in parallel planes possess the same points on Ω.

A *canal surface* is defined as the envelope of a one-parameter family of spheres $\Sigma(t)$ whose centers trace a curve $\mathcal{S}$ in 3-dimensional space parameterized by $\mathbf{s}(t)$ and possess radii $r(t)$ (see Fig. 1), i.e.,

$$\Sigma(t) : \; |(x, y, z)^T - \mathbf{s}(t)|^2 - r(t)^2 = 0. \tag{3}$$

The curve $\mathcal{S}$ is called the *spine curve* and $r(t)$ the *radius function* of the canal surface. For constant $r(t)$ we obtain a *pipe surface*, for $\mathcal{S}$ being a straight line we arrive at a *surface of revolution*.

By appending the corresponding sphere radii $r(t)$ to the points of the spine curve (or the *skeleton*, or the *medial axis*) we obtain the *medial axis transform* (shortly *MAT*), i.e., the curve $\mathcal{M}$: $(x(t), y(t), z(t), r(t))$ in 4-dimensional space. In addition for the canal surface associated to the system of spheres given by $\Sigma(t) : |(x, y, z)^T - \mathbf{s}(t)|^2 - R(t) = 0$, we analogously define the curve $\mathcal{M}^2 : (x(t), y(t), z(t), R(t))$ which is called a *squared medial axis transform* (or shortly *squared MAT*). The rationality of canal surfaces is described by the following proposition, cf. Bastl et al. (2014):

Proposition 1.1. *Any canal surface with the corresponding squared medial axis transform $\mathcal{M}^2$ possesses a rational parameterization if and only if $\mathcal{M}^2$ is a rational curve.*