Journal of Symbolic Computation 74 (2016) 378–399

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Border basis relaxation for polynomial optimization

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ARTICLE INFO

Article history:
Received 23 June 2014
Accepted 31 July 2015
Available online 17 August 2015

Keywords:
Polynomial optimization
Moment matrices
Flat extension
Border basis

ABSTRACT

A relaxation method based on border basis reduction which improves the efficiency of Lasserre's approach is proposed to compute the infimum of a polynomial function on a basic closed semi-algebraic set. A new stopping criterion is given to detect when the relaxation sequence reaches the infimum, using a sparse flat extension criterion. We also provide a new algorithm to reconstruct a finite sum of weighted Dirac measures from a truncated sequence of moments, which can be applied to other sparse reconstruction problems. As an application, we obtain a new algorithm to compute zero-dimensional minimizer ideals and the minimizer points or zero-dimensional G-radical ideals. Experiments show the impact of this new method on significant benchmarks.

1. Introduction

Computing the global infimum of a polynomial function f on a semi-algebraic set is a difficult but important problem, with many applications. A relaxation approach was proposed in Lasserre (2001) (see also Parrilo, 2003; Shor, 1987) which approximates this problem by a sequence of finite dimensional convex optimization problems. These optimization problems can be formulated in terms of linear matrix inequalities on moment matrices associated to the set of monomials of degree $\leq t \in \mathbb{N}$ for increasing values of t. They can be solved by Semi-Definite Programming (SDP) techniques. The sequence of minima converges to the actual infimum f^* of the function under some hypotheses (Lasserre, 2001). In some cases, the sequence even reaches the infimum in a finite number of steps (Laurent, 2007; Nie et al., 2006; Marshall, 2009; Demmel et al., 2007; Ha and Pham, 2010;

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http://dx.doi.org/10.1016/j.jsc.2015.08.004
0747-7171/

Nie, 2011). This approach has proved to be particularly fruitful in many problems (Lasserre, 2009). In contrast with numerical methods such as gradient descent methods, which converge to a local extremum but with no guaranty for the global solution, this relaxation approach can provide certificates for the infimum value f^* in terms of sums of squares representations.

From an algorithmic and computational perspective, however some issues need to be considered.

The size of the SDP problems to be solved is a bottleneck of the method. This size is related to the number of monomials of degree $\leq t$ and increases exponentially with the number of variables and the degree t. Many SDP solvers are based on interior point methods which provide an approximation of the optimal moment sequence within a given precision in a polynomial time: namely $\mathcal{O}((p\,s^{3.5} + c\,p^2 s^{2.5} + c\,p^3 s^{0.5})\log(\epsilon^{-1}))$ arithmetic operations where $\epsilon > 0$ is the precision of the approximation, s is the size of the moment matrices, p is the number of parameters (usually of the order s^2) and c is the number of constraints (Nesterov and Nemirovski, 1994). Thus reducing the size s or the number of parameters p can significantly improve the performance of these relaxation methods. Some recent works address this issue, using symmetries (see e.g. Riener et al., 2013) or polynomial reduction (see e.g. Lasserre et al., 2012). In this paper, we extend this latter approach.

While determining the infimum value of a polynomial function on a semi-algebraic set is important, computing the minimizer points, is also critical in many applications. Determining when and how these minimizer points can be computed from the relaxation sequence is a problem that has been addressed, for instance in Henrion and Lasserre (2005), Nie (2012) using full moment matrices. This approach has been used for solving polynomial equations (Laurent, 2007; Lasserre et al., 2008, 2009; Lasserre, 2009).

The optimization problem can be reformulated as solving polynomial equations related to the (minimal) critical value of the polynomial f on a semi-algebraic set. Polynomial solvers based, for instance, on Gröbner basis or border basis computation can then be used to recover the real critical points from the complex solutions of (zero-dimensional) polynomial systems (see e.g. Parrilo and Sturmfels, 2003; Safey El Din, 2008; Greuet and Safey El Din, 2011). This type of methods relies entirely on polynomial algebra and univariate root finding. So far, there is no clear comparison of these elimination methods and the relaxation approaches.

Contributions. We propose a new method which combines Lasserre's SDP relaxation approach with polynomial algebra, in order to increase the efficiency of the optimization algorithm. Border basis computations are considered for their numerical stability (Mourrain and Trébuchet, 2005, 2008). In principle, any graded normal form technique could be used here.

A new stopping criterion is given to detect when the relaxation sequence reaches the infimum, using a flat extension criterion from Laurent and Mourrain (2009). We also provide a new algorithm to reconstruct a finite sum of weighted Dirac measures from a truncated sequence of moments. This reconstruction method can be used in other problems such as tensor decomposition (Brachat et al., 2010) and multivariate sparse interpolation (Giesbrecht et al., 2009).

As shown in Abril Bucero and Mourrain (2013), Nie et al. (2006), Demmel et al. (2007), Marshall (2009), Nie (2011), Ha and Pham (2010), an exact SDP relaxation can be constructed for "well-posed" optimization problems. As an application, we obtain a new algorithm to compute zero-dimensional minimizer ideals and the minimizer points, or zero-dimensional G-radicals. Experiments show the impact of this new method compared to the previous relaxation constructions.

Content. The paper is organized as follows. Section 2 describes the minimization problem and includes a running example to explain the different steps of our method. In Section 3–5, we describe the ingredients of the main algorithm, which is described in Section 7. In Section 3, we describe the SDP relaxation hierarchies (full moment matrices and border basis). In Section 4, we tackle the sub-problem of how to compute the optimal linear form through the solution of a SDP problem. In Section 5, we tackle the sub-problem of how to verify that we have found the infimum, checking the flat extension property using orthogonal polynomials. In Section 6, we tackle the sub-problem of how to compute the minimizer points using multiplication matrices. Section 7 gives a description of the complete minimization algorithm. Section 8 analyses cases for which an exact relaxation can be constructed. Section 9 concludes experimentation.