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On new types of rational rotation-minimizing frame space curves

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ABSTRACT

The existence of rational rotation-minimizing frames (RRMF) on polynomial space curves is characterized by the satisfaction of a certain identity among rational functions. In this note we prove that previously thought degree limitations on that condition are incorrect. In that regard, new types of RRMF curves are discovered.

1. Introduction

A quaternion polynomial is an expression of the form $\mathcal{A}(t) = u(t) + \mathbf{i}\, v(t) + \mathbf{j}\, p(t) + \mathbf{k}\, q(t)$, where $u(t)$, $v(t)$, $p(t)$, $q(t)$ are polynomials with real coefficients. Polynomials of such nature have enjoyed a wide spread theoretical attention lately in terms of the nature of their roots and factorization over the ring $\mathbb{H}[t]$, where $\mathbb{H}$ is the skew field of quaternions. For a detailed analysis of quaternion polynomials the reader is referred to Gentili and Struppa (2008), Gordon and Motzkin (1965), Topuridze (2009).

A spatial Pythagorean hodograph (PH) curve $\mathbf{r}(t)$ is a polynomial curve whose derivative is generated from a *primitive*[1] quaternion polynomial $\mathcal{A}(t)$ by the product $\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$, where $\mathcal{A}^*(t)$ is the conjugate of $\mathcal{A}(t)$ (Farouki, 2008). An adapted orthonormal frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ on $\mathbf{r}(t)$, where $\mathbf{f}_1$ is the curve tangent, is a rotation-minimizing frame (RMF) if the normal-plane vectors $\mathbf{f}_2$, $\mathbf{f}_3$ exhibit no instantaneous rotation about $\mathbf{f}_1$ (Klok, 1986). For a *rational* RMF it is sufficient and necessary that the condition

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[1] A quaternion polynomial $\mathcal{A}(t) = u(t) + \mathbf{i}\, v(t) + \mathbf{j}\, p(t) + \mathbf{k}\, q(t)$ is said to be *primitive* if $\gcd(u, v, p, q) = 1$. Similarly, a complex polynomial $\alpha(t) + \mathrm{i}\,\beta(t)$ is primitive if $\gcd(\alpha, \beta) = 1$.

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$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{\alpha\beta' - \alpha'\beta}{\alpha^2 + \beta^2} \tag{1}$$

be satisfied for relatively prime polynomials $\alpha(t)$, $\beta(t)$ (Han, 2008). In this case $\mathbf{r}(t)$ is called an RRMF curve. Solutions of (1) defining true space curves were identified in Farouki et al. (2009), Farouki (2010) for $\mathcal{A}(t)$ quadratic, and in Farouki and Sakkalis (2010) for $\mathcal{A}(t)$ of any degree, under the assumption that $\alpha^2 + \beta^2 = u^2 + v^2 + p^2 + q^2$. Moreover, existence of solutions to (1) with $\deg(\alpha^2 + \beta^2) \leq \deg(u^2 + v^2 + p^2 + q^2)$ was established in the case of quintics (Farouki and Sakkalis, 2012). The case where $\deg(\alpha^2 + \beta^2) > \deg(u^2 + v^2 + p^2 + q^2)$ was first considered in Farouki and Sakkalis (2013) since the proof of Part 2 of Remark 5.1 in Farouki and Sakkalis (2010) was incomplete. The latter was restated as follows:

Conjecture 1. *Let $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$ and $\alpha(t) + \mathrm{i}\,\beta(t)$ be primitive, and satisfy (1). Then* $\deg(\alpha^2 + \beta^2) \leq \deg(u^2 + v^2 + p^2 + q^2)$.

The primary purpose of this paper is to show that Conjecture 1 is false by providing an abundance of counterexamples. In view of this, we will define new types of RRMF curves. Finally, we present some results concerning roots, as well as, primality of quaternion polynomials.

2. Preliminaries

This section deals, primarily, with a brief overview of quaternion polynomials and some of their properties pertinent to PH/RRMF curves. For an extensive analysis of quaternion polynomials the reader is referred to Gordon and Motzkin (1965), Topuridze (2009).

Let $\mathbb{H}$ denote the skew field of quaternions. Due to the noncommutative nature of $\mathbb{H}$, polynomials over $\mathbb{H}$ are usually distinguished into the following types: *left*, *right* and *general* (Gordon and Motzkin, 1965). Here, we shall focus on left polynomials, but similar results can be achieved for right ones as well. A (left) quaternion polynomial is an expression of the form

$$\mathcal{A}(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_n \tag{2}$$

where $a_k \in \mathbb{H}$. If $a_0 \neq 0$, n is called the degree of $\mathcal{A}$; moreover, when $a_0 = 1$, $\mathcal{A}$ is called *monic*. An equivalent representation of $\mathcal{A}$, which will be used for the rest of the paper, is $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$, where 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the standard basis elements of $\mathbb{H}$ and $u, v, p, q \in \mathbb{R}[t]$. $\mathcal{A}^*$ is the conjugate of $\mathcal{A}$ and is defined as $\mathcal{A}^* = u(t) - \mathbf{i}\,v(t) - \mathbf{j}\,p(t) - \mathbf{k}\,q(t)$. Let $\mathcal{A}_1(t) = u_1(t) + \mathbf{i}\,v_1(t) + \mathbf{j}\,p_1(t) + \mathbf{k}\,q_1(t)$ be another quaternion polynomial. We define their (formal) product $\mathcal{A}(t) \cdot \mathcal{A}_1(t)$ by

$$\begin{aligned}
&\mathcal{A}(t) \cdot \mathcal{A}_1(t) = U(t) + \mathbf{i}\,V(t) + \mathbf{j}\,P(t) + \mathbf{k}\,Q(t), \quad \text{where} \\
&U = uu_1 - vv_1 - pp_1 - qq_1, \quad V = uv_1 + vu_1 + pq_1 - qp_1 \\
&P = up_1 + pu_1 + qv_1 - vq_1, \quad Q = uq_1 + qu_1 + vp_1 - pv_1.
\end{aligned} \tag{3}$$

In view of the above, we say that $\mathcal{B}(t)$ is a right (left) factor of $\mathcal{A}(t)$ if $\mathcal{A}(t) = \mathcal{A}_2(t) \cdot \mathcal{B}(t)$ ($\mathcal{A}(t) = \mathcal{B}(t) \cdot \mathcal{A}_3(t)$), respectively, where $\mathcal{A}_j(t)$, $j = 2, 3$ are quaternion polynomials.

Now, if $\mathcal{C} \in \mathbb{H}$, we define $\mathcal{A}(\mathcal{C}) = u(\mathcal{C}) + \mathbf{i}\,v(\mathcal{C}) + \mathbf{j}\,p(\mathcal{C}) + \mathbf{k}\,q(\mathcal{C})$; if $\mathcal{A}(\mathcal{C}) = 0$, $\mathcal{C}$ is called a (right) *zero* or a (right) *root* of $\mathcal{A}$. According to Theorem 1 of Gordon and Motzkin (1965) an element $\mathcal{C} \in \mathbb{H}$ is a zero of $\mathcal{A}$ if and only if $(t - \mathcal{C})$ is a right factor of $\mathcal{A}$. In contrast with polynomials over $\mathbb{C}$, quaternion polynomials can have an infinitude of roots. For example, if $g(t) = t^2 + 1$, any quaternion $\mathcal{C} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$ with $\|\mathcal{C}\| = 1$ is a root of g. This, however, cannot happen in the case of primitive polynomials. Indeed, we have

Remark 2.1. Let $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$ be a primitive, monic polynomial of degree $n \geq 1$. Then, if $N(\mathcal{A})$ is the number of roots of $\mathcal{A}(t)$, we have $1 \leq N(\mathcal{A}) \leq n$.