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Algorithms for curves with one place at infinity ☆

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ABSTRACT

Let f be a plane curve. We give a procedure based on Abhyankar's approximate roots to detect if it has a single place at infinity, and if so construct its associated δ-sequence, and consequently its value semigroup. For a δ-sequence we present a procedure to construct all curves having this associated sequence. Also for a fixed genus (equivalently Frobenius number) we construct all δ-sequences generating numerical semigroups with this given genus.

We also study the embeddings of such curves in the plane. In particular, we prove that polynomial curves might not have a unique embedding.

0. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and let $f(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$ be a nonzero polynomial of $\mathbb{K}[x][y]$. Assume, possibly after a change of variables, that $\deg_x(a_i(x)) < i$ for all $1 \le i \le n$. Write $f(x,y) = y^n + \sum_{i,j,i+j<n} c_{ij}x^i y^j$ and let $F(x,y,u) = y^n + \sum c_{ij}u^{n-i-j}x^i y^j \in \mathbb{K}[u,x,y]$. Let C be the curve $f = 0$ in $\mathbb{K}^2$. Then the projective curve $\bar{C} : F = 0$ is the projective closure of C in $\mathbf{P}^2_{\mathbb{K}}$. Furthermore, $p = (0,1,0)$ is the unique point at infinity of $\bar{C}$

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and $F(u,1,y)$ is the local equation of $\bar{C}$ at p. We say that f has one place at infinity if $F(u,1,y)$ is analytically irreducible in $\mathbb{K}[[u,y]]$.

Curves with one place at infinity play an important role in affine geometry. In particular, it has been proved in Abhyankar (1977a) and Abhyankar and Moh (1975) that if f has one place at infinity, then so is for $f-\lambda$ for all $\lambda \in \mathbb{K}$. Also, we can associate with f a numerical semigroup that has some good properties (it is free, and thus a complete intersection and symmetric). By using the arithmetic of the semigroup of a polynomial with one place at infinity and its approximate roots, S.S. Abhyankar and T.T. Moh proved that given two polynomials $x(t), y(t) \in \mathbb{K}[t]$ with t-degrees $n > m$, if $\mathbb{K}[x(t), y(t)] = \mathbb{K}[t]$, then m divides n, showing that a coordinate of $\mathbb{K}^2$ has a unique embedding in $\mathbb{K}^2$ (this result has been generalized to a more general setting by V. Lin and M. Zaidenberg, 1983).

The study of polynomials with one place at infinity is also motivated by the plane Jacobian conjecture. Let $g \in \mathbb{K}[x,y]$. This conjecture says the following: if the Jacobian $J(f,g)$ is a nonzero constant, then $\mathbb{K}[x,y] = \mathbb{K}[f,g]$. If $\mathbb{K}[x,y] = \mathbb{K}[f,g]$, then f is equivalent to a coordinate, in particular f has one place at infinity. Hence the plane Jacobian conjecture is equivalent to the following: if $J(f,g)$ is a nonzero constant, then f has one place at infinity. Despite a lot of activities, this conjecture is still open.

Let f be as above and let g be a polynomial of $\mathbb{K}[x,y]$. Let C_1 be the affine curve $g=0$. We say that C is isomorphic to C_1 if their rings of coordinates $\mathbb{K}[C]$ and $\mathbb{K}[C_1]$ are isomorphic. We say that C is equivalent to C_1 if $f = \sigma(g)$ for some automorphism σ of $\mathbb{K}[x,y]$. It is natural to ask which isomorphic curves are equivalent. Curves with one place at infinity, with their good properties, offer a good setting where this question can be studied.

The main aim of this paper is to give an algorithmic approach to the study of curves with one place at infinity, together with an implementation in GAP (The GAP Group, 2014). Our contributions in this direction are the following. For (2) and (3) we make extensive use of the concept of δ-sequence (see p. 483).

(1) (Section 3) Given a polynomial $f \in \mathbb{K}[x,y]$, decide if f has one place at infinity and if yes, compute the set of its approximate roots and the generators of its semigroup. Our approach here is the irreducibility criterion given by S.S. Abhyankar (1989). This criterion is straightforward from the equation. It is based on the notion of generalized Newton polygons and it does not use neither the resolution of singularities nor the calculation of Puiseux series at infinity. We have implemented this procedure using the `numericalsgps` (Delgado et al., 2015) GAP (The GAP Group, 2014) package, and named it `SemigroupOfValuesOfPlaneCurveWithSinglePlaceAtInfinity`.

(2) (Section 4) Given a sequence of integers in $\mathbb{N}$, decide if this sequence generates the semigroup of a polynomial at infinity, and if it is the case, calculate an equation of such a polynomial. Our equation allows in particular, using the notion of generalized Newton polygons, the calculation of all polynomials with the given semigroup. The GAP function implemented for this is `CurveAssociatedToDeltaSequence`. This algorithm corresponds with Algorithm 1 from Fujimoto and Suzuki (2002). In Fujimoto and Suzuki (2002) the authors are also concerned in the moduli space, which we do not cover here (see also Oka, 1998; Suzuki, 1999). In contrast, we also offer (as mentioned in the preceding paragraph) the semigroup of values and the approximate roots of a plane curve with a single place at infinity, and this is not treated in Fujimoto and Suzuki (2002).

(3) (Section 5) Given a positive integer g, compute the set of semigroups of polynomials with one place at infinity with genus g (hence with conductor $2g$). Note that if Γ is such a semigroup and if f is a polynomial with one place at infinity whose semigroup is Γ, then g is the geometric genus of a nonsingular element of the pencil $(f-\lambda)_{\lambda\in\mathbb{K}}$, and $2g$ is the rank of the $\mathbb{K}$-vector space $\mathbb{K}[x,y]/(f_x, f_y)$, where f_x (respectively f_y) denotes the x-derivative (respectively the y-derivative) of f. The function `DeltaSequencesWithFrobeniusNumber` has been designed to do this. As mentioned in Fujimoto and Suzuki (2002), Suzuki gave also a procedure to compute all δ-sequences with given genus.