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Liouville integrability: An effective Morales–Ramis–Simó theorem

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ABSTRACT

Consider a complex Hamiltonian system and an integral curve. In this paper, we give an effective and efficient procedure to put the variational equation of any order along the integral curve in reduced form provided that the previous one is in reduced form with an abelian Lie algebra. Thus, we obtain an effective way to check the Morales–Ramis–Simó criterion for testing meromorphic Liouville integrability of Hamiltonian systems.

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1. Introduction

Consider a Hamiltonian system of $2n$ differential equations

$$(X_H)\ :\quad \begin{cases} \dot{q}_i &= +\frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{cases}$$

A *first integral* is a function of the q_i and p_i which is constant along the solutions of (X_H). The system is called (meromorphically) *Liouville integrable* (or *completely integrable*) when it admits n (meromorphic) first integrals $F_1, \ldots, F_n$ which are functionally independent (their differentials are linearly independent) and in involution (their Poisson brackets vanish or, equivalently, the associated Hamiltonian vector fields X_{F_i} commute). We refer to the reference books (Abraham and Marsden, 1978; Cushman and Bates, 1997; Audin, 2008) for more on this topic; see also Section 2 for definitions.

The Ziglin–Morales–Ramis theory (see Morales-Ruiz and Ramis, 2010; Audin, 2008 for statements and applications) provides mathematical tools to check when a system is non-integrable. This is particularly useful as Hamiltonian systems generally come as parametrized families. The non-integrability criteria allow one to discard the vast majority of values of the parameters for which the system is not integrable. The principle is as follows. First, we find a particular solution Γ of the system (X_H) (generally from an invariant plane found from symmetries) and we compute variational equations (VE_p), i.e. systems of linear differential equations governing a Taylor expansion of a solution of (X_H) along the particular solution Γ. The Liouville integrability of (X_H) induces integrability conditions on the variational equations (VE_p), which in turn imply properties of their monodromy or differential Galois groups. Technically, the Morales–Ramis–Simó theorem states that if (X_H) is integrable, then the Lie algebras of the differential Galois groups of all variational equations (VE_p) must be abelian (all these terms are defined in Section 2).

The strength of this criterion is that it turns a geometric condition (integrability) into an algebraic one (abelianity of a Lie algebra), thus paving the way for possible computations. However, although there exist general algorithms to compute differential Galois groups of reducible systems such as the variational equations (VE_p) (Feng, 2015; Rettstadt, 2014 or van der Hoeven, 2007), none of them are currently even close to being practical or implemented at this time. Furthermore, the size of the variational equations (VE_p) grows fast, so only a method which uses the structure of the system to make it simpler has a chance of being efficient. The main goal of the present paper is to explain how to use the structure of the system to make it simpler, which will allow us to check efficiently whether its Lie algebra is abelian or not.

Over the past decade, several approaches have been devised to concretely apply this Morales–Ramis–Simó integrability criterion.

For Hamiltonians of the form $H = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + V(q)$, where V is a potential in q, the first variational equation is often a direct sum of Lamé equations of the form $y''(x) = (n(n+1)\wp(x) + B)\, y(x)$, where $\wp$ denotes the Weierstrass function associated to an elliptic curve. In this case, Morales has elaborated a local criterion to find obstructions to integrability on higher variational equations via local computations (see Lemmas 11 and 12 in Morales-Ruiz and Ramis, 2001a, p. 79, and Proposition 7, p. 81). Maciejewski, Przybylska and Duval have elaborated techniques to handle variational equations for the case of Hamiltonians with potentials (Maciejewski and Przybylska, 2006; Duval and Maciejewski, 2009, 2014, 2015); see also the works of Combot and coauthors (Combot, 2013; Combot and Koutschan, 2012; Bostan et al., 2014).

Another approach is to determine numerical trajectories and compute numerical monodromies around these. Although it is difficult to obtain rigorous proofs by these methods, they provide surprisingly precise information. They have been developed, for example, by Martínez and Simó (2009), by Simon and Simó in the Atwood paper (Pujol et al., 2010), by Simon in the more recent (Simon, 2014a, 2014b) and by Salnikov in the paper (Salnikov, 2014, 2013).

The general strategy for turning numerical evidence into rigorous proofs is to show that a certain commutator is non-zero. This in turn yields calculations of integrals and of residues, which can be achieved algorithmically due to their D-finiteness. This is used by Martínez and Simó (2009) and