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Binomial fibers and indispensable binomials ☆

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Let I be an arbitrary ideal generated by binomials. We show that certain equivalence classes of fibers are associated to any minimal binomial generating set of I. We provide a simple and efficient algorithm to compute the indispensable binomials of a binomial ideal from a given generating set of binomials and an algorithm to detect whether a binomial ideal is generated by indispensable binomials.

0. Introduction

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ where $\mathbb{K}$ is a field. A binomial is a polynomial of the form $x^{\mathbf{u}} - \lambda x^{\mathbf{v}}$ where $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ and $\lambda \in \mathbb{K} \setminus \{0\}$, and a binomial ideal is an ideal generated by binomials. We say that the ideal I of R is a *pure binomial ideal* if I is generated by *pure difference binomials*, i.e. binomials of the form $x^{\mathbf{u}} - x^{\mathbf{v}}$ with $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$. Binomial ideals were first studied systematically in Eisenbud and Sturmfels (1996) and this class of ideals also includes lattice ideals. Recall that if $L \subset \mathbb{Z}^n$ is a lattice, then

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the corresponding lattice ideal is defined as $I_L = (x^{\mathbf{u}} - x^{\mathbf{v}} : \ \mathbf{u} - \mathbf{v} \in L)$ and the lattice is saturated exactly when the lattice ideal is toric, i.e. prime. The study of binomial ideals is a rich subject: the classical reference is Sturmfels (1995) and we also refer to Miller (2011) for recent developments. It has applications in various areas in mathematics, such as algebraic statistics, integer programming, graph theory, computational biology, code theory, see Diaconis and Sturmfels (1998), Dück and Zimmermann (2014), Hoşten and Thomas (1998), Ohsugi and Hibi (2005, 2006), Sturmfels and Sullivant (2005), etc.

A particular problem that arises is the *efficient* generation of binomial ideals by a set of binomials. Up to now, it has mainly been addressed for toric and lattice ideals, see Bigatti et al. (1999), Charalambous et al. (2007, 2013), Hemmecke and Malkin (2009), Hoşten and Sturmfels (1995), Sturmfels (1995) among others. In Section 1, we consider this problem in the case of binomial ideals. For this we study the *fibers* of binomial ideals: in Dickenstein et al. (2010, Proposition 2.4) an equivalence relation on $\mathbb{N}^n$ was introduced for any binomial ideal I of R, namely $\mathbf{u} \sim_I \mathbf{v}$ if $x^{\mathbf{u}} - \lambda x^{\mathbf{v}} \in I$ for some $\lambda \neq 0$. For each such equivalence class, we get a *fiber* on the set of monomials: the I-fiber of $x^{\mathbf{u}}$ is the set $\{x^{\mathbf{v}} : \ \mathbf{u} \sim_I \mathbf{v}\}$. When $I := I_L$ is the lattice ideal of L, the equivalence class of $\mathbf{u}$ consists precisely of all $\mathbf{v}$ such that $\mathbf{u} - \mathbf{v} \in L$ and the I-fibers are finite exactly when $L \cap \mathbb{N}^n = \{\mathbf{0}\}$. In this case, for each I-fiber one can use a graph construction, see Diaconis and Sturmfels (1998), Charalambous et al. (2007), that determines the I-fibers that appear as invariants associated to any minimal generating set of I. We also note that in Charalambous et al. (2013) the fibers of I_L were studied even when $L \cap \mathbb{N}^n \neq \{\mathbf{0}\}$. In all cases finite or not, it is clear that divisibility of monomials does not induce necessarily a meaningful partial order on the set of I-fibers. In this paper for any binomial ideal I we define an equivalence relation on the set of I-fibers and then order the equivalence classes of I-fibers, see Definition 1.6. We note that this was first done in Charalambous et al. (2013) for the case $I = I_L$. This partial order allows us to prove that a certain set of equivalence classes of I-fibers is an invariant, associated to any generating set of I, see Theorem 1.12. We note that lattice ideals have all fibers either finite or infinite and the equivalence classes of fibers for lattice ideals have the same cardinality, see Charalambous et al. (2013, Propositions 2.3 and 3.5). However this might not be the case for a binomial ideal and this constitutes an added degree of difficulty, see Examples 1.2 c) and 1.7 c). Moreover we show that binomial ideals which contain monomials have a unique maximal fiber consisting of all monomials of I, see Theorem 1.8.

A related question that attracted a lot of interest in the recent years is whether there is a unique minimal binomial generating set for a binomial ideal. One of the first papers to deal with this question for lattice ideals from a purely theoretical point of view was Peeva and Sturmfels (1998). As it turns out, the positive answer has applications to Algebraic Statistics: Aoki and Takemura (2003), Aoki et al. (2008), Ohsugi and Hibi (2005). Thus in Ohsugi and Hibi (2006) and Aoki et al. (2008) the notions of *indispensable* monomials and binomials were defined. Let I be a binomial ideal. A binomial is called *indispensable* if (up to a nonzero constant) it belongs to every minimal generating set of I consisting of binomials. This implies of course that (up to a nonzero constant) it belongs to every binomial generating set of I. A monomial is called *indispensable* if it is a monomial term of at least one binomial in every system of binomial generators of I. How does one compute these elements?

When $I := I_L$ is a lattice ideal and $L \cap \mathbb{N}^n = \{\mathbf{0}\}$, there are several works in the literature that deal with this problem. In particular, in Ohsugi and Hibi (2006) it was shown that to compute the indispensable binomials of I_L, one computes all lexicographic reduced Gröbner bases and then their intersection: there are $n!$ such bases; a corresponding result for indispensable monomials was shown in Aoki et al. (2008). In Ojeda and Vigneron-Tenorio (2010), it was shown that to compute the indispensable binomials of I_L, it is enough to compute certain degree-reverse lexicographic reduced Gröbner bases of I_L (n of them), and then compute their intersection. In Charalambous et al. (2007, Proposition 3.1), it was shown that to find the indispensable monomials of I_L, it is enough to consider any one of the binomial generating sets of I_L. Moreover in Charalambous et al. (2007, Theorem 2.12) it was shown that in order to find the indispensable binomials of I_L, it is enough to consider any minimal binomial generating set of I_L, assign $\mathbb{Z}^n/L$-degrees to the binomials of this set and to compute their minimal $\mathbb{Z}^n/L$-degrees. More recently in Katsabekis and Ojeda (2014, Theorem 1.1, Corollary 1.3), it was shown that if I is a pure binomial ideal then there is a $d \in \mathbb{N}$ such that any I is A-graded for some $A \subset \mathbb{Z}^d$: when $\mathbb{N}A \cap (-\mathbb{N}A) = \{\mathbf{0}\}$ and all fibers are finite a sufficient condition was given in