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Deciding positivity of multisymmetric polynomials

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ABSTRACT

The question how to certify non-negativity of a polynomial function lies at the heart of Real Algebra and also has important applications to Optimization. In this article we investigate the question of non-negativity in the context of multisymmetric polynomials. In this setting we generalize the characterization of non-negative symmetric polynomials given in Timofte (2003), Riener (2012) by adapting the method of proof developed in Riener (2013). One particular case where our results can be applied is the question of certifying that a (multi-)symmetric polynomial defines a convex function. As a direct corollary of our main result we deduce that in the case of a fixed degree it is possible to derive a method to test for convexity which makes use of the special structure of (multi-)symmetric polynomials. In particular it follows that we are able to drastically simplify the algorithmic complexity of this question in the presence of symmetry. This is not to be expected in the general (i.e. non-symmetric) case, where it is known that testing for convexity is NP-hard already in the case of polynomials of degree 4 (Ahmadi et al., 2013).

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1. Introduction

A real polynomial is called positive (non-negative) if its evaluation on every real point is positive (non-negative). The study of this property of polynomial functions is one of the aspects that separates Real Algebraic Geometry from Algebraic Geometry over algebraically closed fields. Indeed, Real Algebraic Geometry developed building on Hilbert's problem of characterizing non-negative polynomials via sums of squares. On the complexity side, it is known that the problem to algorithmically decide whether a given polynomial assumes only positive or non-negative values is NP-hard in general (see for example Murty and Kabadi, 1987; Blum et al., 1997) and is essential for example to understand global optimization of polynomial functions. Besides the general results some authors have studied particular cases of polynomials which are invariant under group actions, for example by permuting the variables. In particular, symmetric polynomials, i.e. polynomials invariant under all permutations of the variables, exhibit some interesting properties that behave differently over real closed and algebraically closed fields. For example, Basu and Riener (2013) show, that the number of connected components of the orbit space of a complex variety which is defined by symmetric polynomials of a given maximal degree is bounded by a quantity which (for a large number of variables) only depends on this maximal degree. In contrast to that there are examples of real varieties where this does not hold, i.e., this quantity actually grows with the number of variables. Therefore, the geometry of symmetric real varieties and semi-algebraic sets can be much more complicated.

Combining Artin's solution to Hilbert's 17th problem with Hermite's quadratic form, which characterizes real univariate polynomials with only real roots, Procesi (1978) was able to give a *Positivstellensatz* characterizing all symmetric non-negative polynomials. However, this characterization is not very advantageous in situations, where the degree of the polynomials is much smaller than the number of variables. For this situations, Timofte (2003) was able to provide a characterization of symmetric non-negative functions of a fixed degree that can be used to algorithmically certify non-negativity: He could establish that a symmetric polynomial of degree $2d$ is non-negative if and only if it is non-negative on all points with at most d distinct coordinates. This observation, which generalized an earlier statement by Harris (1999), leads to a number of interesting consequences. For example, it provides an essential part to the description of the asymptotical behavior of the cone of symmetric non-negative forms of a given degree when the number of variables grows (Blekherman and Riener, 2012). Algorithmically, this result allows to show that the complexity of deciding non-negativity of a symmetric function with a fixed degree only grows polynomially in the number of variables. Following the work of Timofte, the second author was able to provide short proofs of this characterization of symmetric non-negative polynomials (Riener, 2012; Riener, 2013).

Contributions: In this article we extend the previous results on symmetric polynomials to arrive at a similar characterization of multisymmetric polynomial functions that assume only positive (non-negative) values. The class of multisymmetric polynomials naturally generalizes the symmetric polynomials. Whereas symmetric polynomials are invariant by all permutations of the variables, multisymmetric polynomials can be thought of as functions which are invariant under simultaneously permuting k-tuples of variables. Similar to the case of symmetric polynomials, we are able to show that when the degree of such a polynomial is sufficiently smaller than the number of variables, also for these multisymmetric polynomials non-negativity can be checked on a lower-dimensional subset consisting of points whose orbit length is not maximal. As in the case of the usual action of the symmetric group, these points lie on linear subspaces.

Our main result is Theorem 14 which bounds the dimension of subspaces one has to consider to decide non-negativity of a multisymmetric polynomial. Besides this general bound the idea of the proof can be adjusted in particular situations to derive stronger bounds. We give several other bounds to illustrate this.

As an application of our results we investigate the question of deciding if a given symmetric or multisymmetric polynomial defines a convex function. It is straightforward to observe that the question of convexity of a k-symmetric polynomial in kn many variables leads to the question of certifying