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Desingularization of Ore operators

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ABSTRACT

We show that Ore operators can be desingularized by calculating a least common left multiple with a random operator of appropriate order, thereby turning a heuristic used for many years in several computer algebra systems into an algorithm. Our result can be viewed as a generalization of a classical result about apparent singularities of linear differential equations.

1. Introduction

Consider a linear ordinary differential equation, like for example

$$x(1-x)f'(x) - f(x) = 0.$$

The leading coefficient polynomial $x(1-x)$ of the equation is of special interest because every point ξ which is a singularity of some solution of the differential equation is also a root of this polynomial. However, the converse is in general not true. In the example above, the root $\xi = 1$ indicates the singularity of the solution $x/(1-x)$, but there is no solution which has a singularity at the other root $\xi = 0$. To see this, observe that after differentiating the equation, we can cancel ("remove") the factor x from it. The result is the higher order equation

$$(1-x)f''(x) - 2f'(x) = 0,$$

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whose solution space contains the solution space of the original equation. Such a calculation is called *desingularization*. The factor x is said to be *removable*.

Given a differential equation, it is of interest to decide which factors of its leading coefficient polynomial are removable, and to construct a higher order equation in which all the removable factors are removed. A classical algorithm, which is known since the end of the 19th century (Schlesinger, 1895; Ince, 1926), proceeds by taking the least common left multiple of the given differential operator with a suitably constructed auxiliary operator. This algorithm is summarized in Section 2 below. At the end of the 20th century, the corresponding problem for linear recurrence equations was studied and algorithms for identifying removable factors have been found and their relations to "singularities" of solutions have been investigated (Abramov and van Hoeij, 1999, 2003; Abramov et al., 2006). Also some steps towards a unified theory for desingularization of Ore operators have been made (Chyzak et al., in preparation; Chen et al., 2013). Possible connections to Ore closures of an operator ideal have been noted in Chyzak et al. (in preparation) and within the context of order-degree curves (Chen et al., 2013, Chen and Kauers, 2012a; 2012b). These will be further developed in a future paper.

Our contribution in the present article is a three-fold generalization of the classical desingularization algorithm for differential equations. Our main result (Theorem 6 below) says that (a) instead of the particular auxiliary operator traditionally used, almost every other operator of appropriate order also does the job, (b) also the case is covered where a multiple root can't be removed completely but only its multiplicity can be reduced, and (c) the technique works not only for differential operators but for every Ore algebra. Code fragments in the Maple library (e.g., the function DEtools/Homomorphisms/AppCheck) indicate that some people have already observed before us that taking lclm with a random operator tends to remove removable factors and used this as a heuristic. We give here for the first time a rigorous justification of this phenomenon.

For every removable factor p there is a smallest $n \in \mathbb{N}$ such that removing p from the operator requires increasing the order of the operator by at least n. Classical desingularization algorithms compute for each factor p an upper bound for this n, and then determine whether or not it is possible to remove p at the cost of increasing the order of the operator by at most n. In the present paper, we do not address the question of finding bounds on n but only discuss the second part: assuming some $n \in \mathbb{N}$ is given as part of the input, we consider the task of removing as many factors as possible without increasing the order of the operator by more than n. Of course, for Ore algebras where it is known how to obtain bounds on n, these bounds can be combined with our result.

Recall the notion of Ore algebras (Ore, 1933). Let K be a field of characteristic zero. Let $\sigma: K[x] \to K[x]$ be a ring automorphism that leaves the elements of K fixed, and let $\delta: K[x] \to K[x]$ be a K-linear map satisfying the law $\delta(uv) = \delta(u)v + \sigma(u)\delta(v)$ for all $u, v \in K[x]$. The algebra $K[x][\partial]$ consists of all polynomials in ∂ with coefficients in $K[x]$ together with the usual addition and the unique (in general noncommutative) multiplication satisfying $\partial u = \sigma(u)\partial + \delta(u)$ for all $u \in K[x]$ is called an *Ore algebra*. The field K is called the *constant field* of the algebra. Every nonzero element L of an Ore algebra $K[x][\partial]$ can be written uniquely in the form

$$L = \ell_0 + \ell_1\partial + \cdots + \ell_r\partial^r$$

with $\ell_0, \ldots, \ell_r \in K[x]$ and $\ell_r \neq 0$. We call $\deg_\partial(L) := r$ the *order* of L and $\mathrm{lc}_\partial(L) := \ell_r$ the *leading coefficient* of L. Roots of the leading coefficient ℓ_r are called singularities of L. Prominent examples of Ore algebras are the algebra of linear differential operators (with $\sigma = \mathrm{id}$ and $\delta = \frac{d}{dx}$; we will write D instead of ∂ in this case) and the algebra of linear recurrence operators (with $\sigma(x) = x + 1$ and $\delta = 0$; we will write S instead of ∂ in this case).

We shall suppose that the reader is familiar with these definitions and facts, and will make free use of well-known facts about Ore algebras, as explained, for instance, in Ore (1933), Bronstein and Petkovšek (1996), Abramov et al. (2005). In particular, we will make use of the notion of least common left multiples (lclm) of elements of Ore algebras: $L \in K(x)[\partial]$ is a *common left multiple* of $P, Q \in K(x)[\partial]$ if we have $L = UP = VQ$ for some $U, V \in K(x)[\partial]$, it is called a *least common left multiple* if there is no common left multiple of lower order. Least common left multiples are unique up to left-multiplication by nonzero elements of $K(x)$. By $\mathrm{lclm}(P, Q)$ we denote a least common left multiple whose coefficients belong to $K[x]$ and share no common divisors in $K[x]$. Note that $\mathrm{lclm}(P, Q)$ is