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Syzygies of Jacobian ideals and weighted homogeneous singularities

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ABSTRACT

Let V be a projective hypersurface having only isolated singularities. We show that these singularities are weighted homogeneous if and only if the Koszul syzygies among the partial derivatives of an equation for V are exactly the syzygies with a generic first component vanishing on the singular locus subscheme of V. This yields in particular a positive answer in this setting to a question raised by Morihiko Saito and the first author. Finally we explain how our result can be used to improve the listing of Jacobian syzygies of a given degree by a computer algebra system such as Singular, CoCoA or Macaulay2.

1. Introduction and statement of results

Let X be the complex projective space $\mathbb{P}^n$ and consider the associated graded polynomial algebra $S = \oplus_k S_k$ with $S_k = H^0(X, \mathcal{O}_X(k))$. For a nonzero section $f \in S_N$ with $N > 1$, we consider the hypersurface $V = V(f)$ in X given by the zero locus of f and let Y denote the singular locus of V, endowed with its natural scheme structure, see Dimca (2013). We assume in this paper that V has isolated singularities.

Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the ideal sheaf defining this subscheme $Y \subset X$ and consider the graded ideal $I = \oplus_k I_k$ in S with $I_k = H^0(X, \mathcal{I}_Y(k))$. Let $Z = Spec(S)$ be the corresponding affine space $\mathbb{A}^{n+1}$ and denote by $\Omega^j = H^0(Z, \Omega^j_Z)$ the S-module of global, regular j-forms on Z. Using a linear coordinate system $x = (x_0, \ldots, x_n)$ on X, one sees that there is a natural isomorphism of S-modules

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$$\Omega^j = S^{\binom{n+1}{j}} \tag{1.1}$$

which is used to put a grading (independent of the choice of x) on the modules Ω^j, i.e. a differential form

$$\omega = \sum_K \omega_K(x) dx_K \tag{1.2}$$

is homogeneous of degree m if all the coefficients $\omega_K(x)$ are in S_m for all multi-indices $K = (k_1 < \ldots < k_j)$ and $dx_K = dx_{k_1} \wedge \ldots \wedge dx_{k_j}$.

Since f can be thought of as a homogeneous polynomial of degree N on Z, it follows that there is a well defined differential 1-form $df \in \Omega^1$. Using this we define two graded S-submodules in Ω^n, namely

$$AR(f) = \ker\{df \wedge : \Omega^n \to \Omega^{n+1}\} \tag{1.3}$$

and

$$KR(f) = \mathrm{im}\{df \wedge : \Omega^{n-1} \to \Omega^n\}. \tag{1.4}$$

If one computes in a coordinate system x, then $AR(f)$ is the module of *all relations* of the type

$$R : a_0 f_{x_0} + \ldots + a_n f_{x_n} = 0, \tag{1.5}$$

with f_{x_j} being the partial derivative of the polynomial f with respect to x_j. Moreover, $KR(f)$ is the module of *Koszul relations* spanned by obvious relations of the type $f_{x_j} f_{x_i} + (-f_{x_i}) f_{x_j} = 0$ and the quotient

$$ER(f) = AR(f)/KR(f) \tag{1.6}$$

is the module of *essential relations* (which is of course nothing else but the n-th cohomology group of the Koszul complex of $f_{x_0}, \ldots, f_{x_n}$), see Dimca (2013), Dimca and Sticlaru (2015). Note also that with this notation, the ideal I is just the saturation of the Jacobian ideal $J_f = (f_{x_0}, \ldots, f_{x_n})$. In particular, if V is a nodal hypersurface, then I is just the radical of J_f and hence $h \in I$ if and only if h vanishes at all the nodes in Y, see Dimca (2013).

It is easy to compute the dimension of the spaces $AR(f)_m$, $KR(f)_m$ and $ER(f)_m$ in terms of the dimensions of the spaces $M(f)_s$, where $M(f) = S/J_f$ is the Milnor algebra of the degree N hypersurface $V : f = 0$. Indeed one has the following easy result, whose proof is left to the reader. For the last claim one may see also formula (2.17) in Dimca and Sticlaru (2015).

Proposition 1.1. *Let $g \in S_N$ be a generic polynomial (such that $g = 0$ is a smooth hypersurface). Then one has the following.*

(1) $\dim AR(f)_m = (n+1)\binom{n+m}{n} - \binom{n+m+N-1}{n} + \dim M(f)_{m+N-1}$.
(2) $\dim KR(f)_m = (n+1)\binom{n+m}{n} - \binom{n+m+N-1}{n} + \dim M(g)_{m+N-1}$. *In particular, this number depends only on n and N.*
(3) $\dim ER(f)_m = \dim M(f)_{m+N-1} - \dim M(g)_{m+N-1}$.

The main result of this note is the following characterization of the fact that the singularities of V are all weighted homogeneous.

Theorem 1.2. *Assume that the coordinates x have been chosen such that the hyperplane $H_0 : x_0 = 0$ is transversal to V. Consider the projection on the first factor*

$$p_0 : AR(f)_m \to S_m/I_m, \quad (a_0, \ldots, a_n) \mapsto [a_0]. \tag{1.7}$$

Then the equality