Journal of Symbolic Computation 73 (2016) 87–103

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Geometry of the ringed surfaces in $\mathbb{R}^4$ that generate spatial Pythagorean hodographs

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ARTICLE INFO

Article history:
Received 24 September 2014
Accepted 12 March 2015
Available online 17 March 2015

Keywords:
Pythagorean-hodograph curves
Quaternion polynomials
Hopf map
Four-dimensional geometry
Ringed surfaces
Gaussian curvature
Rotation-minimizing frames

ABSTRACT

A *Pythagorean-hodograph* (PH) *curve* $\mathbf{r}(t) = (x(t), y(t), z(t))$ has the distinctive property that the components of its derivative $\mathbf{r}'(t)$ satisfy $x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t)$ for some polynomial $\sigma(t)$. Consequently, the PH curves admit many exact computations that otherwise require approximations. The Pythagorean structure is achieved by specifying $x'(t), y'(t), z'(t)$ in terms of polynomials $u(t), v(t), p(t), q(t)$ through a construct that can be interpreted as a mapping from $\mathbb{R}^4$ to $\mathbb{R}^3$ defined by a quaternion product or the Hopf map. Under this map, $\mathbf{r}'(t)$ is the image of a ringed surface $\mathcal{S}(t, \phi)$ in $\mathbb{R}^4$, whose geometrical properties are investigated herein. The generation of $\mathcal{S}(t, \phi)$ through a family of four-dimensional rotations of a "base curve" is described, and the first fundamental form, Gaussian curvature, total area, and total curvature of $\mathcal{S}(t, \phi)$ are derived. Furthermore, if $\mathbf{r}'(t)$ is non-degenerate, $\mathcal{S}(t, \phi)$ is not developable (a non-trivial fact in $\mathbb{R}^4$). It is also shown that the pre-images of spatial PH curves equipped with a *rotation-minimizing orthonormal frame* (comprising the tangent and normal-plane vectors with no instantaneous rotation about the tangent) are *geodesics* on the surface $\mathcal{S}(t, \phi)$. Finally, a geometrical interpretation of the algebraic condition characterizing the simplest non-trivial instances of *rational* rotation-minimizing frames on polynomial space curves is derived.

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http://dx.doi.org/10.1016/j.jsc.2015.03.005
0747-7171/

1. Introduction

The *parametric speed* $\sigma(t) = |\mathbf{r}'(t)|$ of a polynomial curve $\mathbf{r}(t)$ in $\mathbb{R}^3$ specifies the derivative $\mathrm{d}s/\mathrm{d}t$ of arc length s with respect to the parameter t. Since $\sigma(t)$ is, in general, the square root of a polynomial, the tangent and curvature are not rational in t, and the arc length does not admit exact computation. The inexact nature of such basic properties of a curved locus is problematic for computer-aided design, computer graphics, animation, motion planning for robotics, manufacturing, inspection, and related applications.

The *Pythagorean-hodograph* (PH) *curves* circumvent these limitations by incorporating a Pythagorean constraint on the curve derivative components (Farouki, 2008). For planar curves, this is achieved (Farouki, 1994) through a complex-variable model. For space curves, models based upon quaternions or the *Hopf map* (Hopf, 1931) from $\mathbb{R}^4$ to $\mathbb{R}^3$ are required (Choi et al., 2002). In either case, the pre-image of $\mathbf{r}'(t)$ is actually a (two-dimensional) surface $\mathcal{S}(t,\phi)$ in $\mathbb{R}^4$, that can be parameterized in terms of the variable $t \in [\,0,1\,]$ and an angular variable $\phi \in [\,0,2\pi\,]$.

An adapted orthonormal frame $(\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t))$ on a space curve $\mathbf{r}(t)$ is such that $\mathbf{f}_1(t)$ is the curve tangent while $\mathbf{f}_2(t), \mathbf{f}_3(t)$ span the normal plane at each point. Any curve on the surface $\mathcal{S}(t,\phi)$ between $t = 0$ and 1 serves as a pre-image for $\mathbf{r}'(t)$, but different pre-image curves generate frames with distinct normal-plane vectors. However, a rotation-minimizing frame (RMF) or Bishop frame (Bishop, 1975) is characterized by a particular geometry of the pre-image curve on $\mathcal{S}(t,\phi)$. The RMF maintains a zero angular velocity component in the tangent direction, and is useful in rigid-body motion planning, computer animation, computer numerical control of multi-axis machines, and related applications. The identification of PH curves that admit *rational* RMFs has recently been the subject of considerable interest (Choi and Han, 2002; Farouki, 2010; Farouki et al., 2012; Farouki and Sakkalis (2010, 2012)).

The geometry of the surface $\mathcal{S}(t,\phi)$ is studied herein, and key properties are determined. Expressions for the Gaussian curvature, total area, and total curvature are obtained, and the surface is shown to be non-developable when $\mathbf{r}(t)$ is true space curve. The pre-images of PH curves equipped with RMFs are characterized as geodesics on $\mathcal{S}(t,\phi)$, and a geometrical interpretation in $\mathbb{R}^4$ of the quaternion constraint identifying quintic PH curves with rational RMFs is presented. These results may offer insight into other open problems, such as the choice of free variables that arise in algorithms for constructing spatial PH curves from discrete geometrical data (Farouki et al., 2008), and the generalization of known conditions for rational RMFs on quintic PH curves (Farouki, 2010).

This paper is organized as follows. After briefly reviewing the definitions of spatial PH curves in Section 2, an expression for the Hopf map pre-image circle in $\mathbb{R}^4$ of a point in $\mathbb{R}^3$ is derived in Section 3, and used to formulate the pre-image surface $\mathcal{S}(t,\phi)$ of a given Pythagorean hodograph $\mathbf{r}'(t)$. The structure of this surface is interpreted in Section 4, in terms of rotations of a “base curve” in $\mathbb{R}^4$, and key properties — the first fundamental form, Gaussian curvature, total area, and total curvature — are derived in Section 5. It is also shown that $\mathcal{S}(t,\phi)$ is non-developable if $\mathbf{r}'(t)$ is non-degenerate (developables in $\mathbb{R}^4$ are not necessarily ruled surfaces). The geometry of curves on the surface $\mathcal{S}(t,\phi)$ is then studied in Section 6, and it is observed that RMFs on PH curves are generated by geodesic curves. The existence of *rational* RMFs is addressed in Section 7, and a geometrical interpretation of the constraint that identifies the simplest (quintic) examples of such curves is presented. Finally, Section 8 summarizes the main results of this study, and identifies some problems that deserve further investigation.

2. Pythagorean-hodograph curves

A polynomial *Pythagorean-hodograph* (PH) *curve* $\mathbf{r}(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$ is characterized (Farouki, 2008) by the property that the components of its derivative $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ satisfy the Pythagorean condition

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \tag{1}$$

for some polynomial $\sigma(t)$. A PH curve may be generated from a quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ by integrating the product