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Solving a sparse system using linear algebra

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ABSTRACT

We give a new theoretical tool to solve sparse systems with finitely many solutions. It is based on toric varieties and basic linear algebra; eigenvalues, eigenvectors and coefficient matrices. We adapt Eigenvalue theorem and Eigenvector theorem to work with a canonical rectangular matrix (the first Koszul map) and prove that these new theorems serve to solve overdetermined sparse systems and to count the expected number of solutions.

0. Introduction

0.1. Overview of the problem

In this article we generalize two methods to solve systems of polynomial equations using a coefficient matrix. One method is based on the eigenvalue theorem, first noticed in Lazard (1981). Another, on the eigenvector theorem, first described in Auzinger and Stetter (1988). Let us start describing them.

For simplicity, consider a generic system of n polynomial equations with finitely many solutions in $\mathbb{C}^n$, all with multiplicity one,

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$$\begin{cases} f_1(x_1,\dots,x_n) = 0 \\ \vdots \\ f_n(x_1,\dots,x_n) = 0 \end{cases}$$

where $f_1,\dots,f_n$ are polynomials in $\mathbb{C}[x_1,\dots,x_n]$. The quotient ring,

$$\mathcal{R} = \mathbb{C}[x_1,\dots,x_n]/\langle f_1,\dots,f_n\rangle,$$

is a finite-dimensional vector space and its dimension is the number of solutions (we are assuming that all the solutions have multiplicity one).

Every polynomial $f \in \mathbb{C}[x_1,\dots,x_n]$, determines a linear map $M_f : \mathcal{R} \to \mathcal{R}$,

$$M_f(\bar{g}) = \overline{fg}, \quad g \in \mathbb{C}[x_1,\dots,x_n],$$

where $\bar{g}$ denotes the class of the polynomial g in the quotient ring $\mathcal{R}$. The matrix of M_f is called the *multiplication matrix* associated to the polynomial f.

Theorem *(Eigenvalue Theorem). The eigenvalues of M_f are $\{f(\xi_1),\dots,f(\xi_r)\}$, where $\{\xi_1,\dots,\xi_r\}$ are the solutions of the system of polynomial equations. See Dickenstein and Emiris (2005, Theorem 2.1.4) for a proof.*

Theorem *(Eigenvector Theorem). Let $f = \alpha_1 x_1 + \dots + \alpha_n x_n$ be a generic linear form and let M_f be its multiplication matrix. Assume that $B = \{1, x_1, \dots, x_n, \dots\}$ is a finite basis of $\mathcal{R}$ formed by monomials. Then the left eigenvectors of M_f determine all the solutions of the system of polynomial equations. Specifically, if $v = (v_0,\dots,v_n,\dots)$ is a left eigenvector of M_f such that $v_0 = 1$, then $(v_1,\dots,v_n)$ is a solution of the system of polynomial equations. See Dickenstein and Emiris (2005, §2.1.3) for a proof.*

Now, let us describe the construction of the coefficient matrix (also in the case of polynomial equations).

Let $d = d_1 + \dots + d_n - n + 1$, where $d_i = \deg(f_i)$, $1 \le i \le n$. Let S_d be the space of polynomials of degree $\le d$. Consider the following sets of monomials,

$$\begin{aligned} B_n &= \{x_1^{m_1}\dots x_n^{m_n} \in S_d : d_n \le m_n\} \\ B_{n-1} &= \{x_1^{m_1}\dots x_n^{m_n} \in S_d \setminus B_n : d_{n-1} \le m_{n-1}\} \\ &\vdots \\ B_1 &= \{x_1^{m_1}\dots x_n^{m_n} \in S_d \setminus B_2 : d_1 \le m_1\} \\ B_0 &= \{x_1^{m_1}\dots x_n^{m_n} \in S_d \setminus B_1\}. \end{aligned}$$

Using these sets, we can consider the following linear map,

$$\Psi : \langle B_0\rangle \times \dots \times \langle B_n\rangle \to S_d, \quad \Psi(g_0,\dots,g_n) = f_0 \cdot g_0 + \sum_{i=1}^{n} f_i \cdot g_i,$$

where the polynomial f_0 is a generic linear form and $\langle B_i\rangle$ is the vector space generated by B_i, $0 \le i \le n$. The *coefficient matrix* M is the matrix of Ψ in the monomial bases $B_0,\dots,B_n$. It is a square matrix and can be divided into four blocks,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

The relation between the coefficient matrix and the multiplication matrix is the following,