

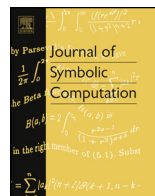


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# Solving a sparse system using linear algebra

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## ARTICLE INFO

### Article history:

Received 11 May 2014

Accepted 12 May 2015

Available online 12 June 2015

### MSC:

14M25

13P15

### Keywords:

Multiplication matrix

Eigenvector

Sparse system

Toric varieties

## ABSTRACT

We give a new theoretical tool to solve sparse systems with finitely many solutions. It is based on toric varieties and basic linear algebra; eigenvalues, eigenvectors and coefficient matrices. We adapt Eigenvalue theorem and Eigenvector theorem to work with a canonical rectangular matrix (the first Koszul map) and prove that these new theorems serve to solve overdetermined sparse systems and to count the expected number of solutions.

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## 0. Introduction

### 0.1. Overview of the problem

In this article we generalize two methods to solve systems of polynomial equations using a coefficient matrix. One method is based on the eigenvalue theorem, first noticed in [Lazard \(1981\)](#). Another, on the eigenvector theorem, first described in [Auzinger and Stetter \(1988\)](#). Let us start describing them.

For simplicity, consider a generic system of  $n$  polynomial equations with finitely many solutions in  $\mathbb{C}^n$ , all with multiplicity one,

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<sup>2</sup> The author was fully supported by CONICET, Argentina.

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_n(x_1, \dots, x_n) = 0 \end{cases}$$

where  $f_1, \dots, f_n$  are polynomials in  $\mathbb{C}[x_1, \dots, x_n]$ . The quotient ring,

$$\mathcal{R} = \mathbb{C}[x_1, \dots, x_n] / \langle f_1, \dots, f_n \rangle,$$

is a finite-dimensional vector space and its dimension is the number of solutions (we are assuming that all the solutions have multiplicity one).

Every polynomial  $f \in \mathbb{C}[x_1, \dots, x_n]$ , determines a linear map  $M_f : \mathcal{R} \rightarrow \mathcal{R}$ ,

$$M_f(\bar{g}) = \overline{fg}, \quad g \in \mathbb{C}[x_1, \dots, x_n],$$

where  $\bar{g}$  denotes the class of the polynomial  $g$  in the quotient ring  $\mathcal{R}$ . The matrix of  $M_f$  is called the *multiplication matrix* associated to the polynomial  $f$ .

**Theorem (Eigenvalue Theorem).** *The eigenvalues of  $M_f$  are  $\{f(\xi_1), \dots, f(\xi_r)\}$ , where  $\{\xi_1, \dots, \xi_r\}$  are the solutions of the system of polynomial equations. See [Dickenstein and Emiris \(2005, Theorem 2.1.4\)](#) for a proof.*

**Theorem (Eigenvector Theorem).** *Let  $f = \alpha_1 x_1 + \dots + \alpha_n x_n$  be a generic linear form and let  $M_f$  be its multiplication matrix. Assume that  $B = \{1, x_1, \dots, x_n, \dots\}$  is a finite basis of  $\mathcal{R}$  formed by monomials. Then the left eigenvectors of  $M_f$  determine all the solutions of the system of polynomial equations. Specifically, if  $v = (v_0, \dots, v_n, \dots)$  is a left eigenvector of  $M_f$  such that  $v_0 = 1$ , then  $(v_1, \dots, v_n)$  is a solution of the system of polynomial equations. See [Dickenstein and Emiris \(2005, §2.1.3\)](#) for a proof.*

Now, let us describe the construction of the coefficient matrix (also in the case of polynomial equations).

Let  $d = d_1 + \dots + d_n - n + 1$ , where  $d_i = \deg(f_i)$ ,  $1 \leq i \leq n$ . Let  $S_d$  be the space of polynomials of degree  $\leq d$ . Consider the following sets of monomials,

$$\begin{aligned} B_n &= \{x_1^{m_1} \dots x_n^{m_n} \in S_d : d_n \leq m_n\} \\ B_{n-1} &= \{x_1^{m_1} \dots x_n^{m_n} \in S_d \setminus B_n : d_{n-1} \leq m_{n-1}\} \\ &\vdots \\ B_1 &= \{x_1^{m_1} \dots x_n^{m_n} \in S_d \setminus B_2 : d_1 \leq m_1\} \\ B_0 &= \{x_1^{m_1} \dots x_n^{m_n} \in S_d \setminus B_1\}. \end{aligned}$$

Using these sets, we can consider the following linear map,

$$\Psi : \langle B_0 \rangle \times \dots \times \langle B_n \rangle \rightarrow S_d, \quad \Psi(g_0, \dots, g_n) = f_0 \cdot g_0 + \sum_{i=1}^n f_i \cdot g_i,$$

where the polynomial  $f_0$  is a generic linear form and  $\langle B_i \rangle$  is the vector space generated by  $B_i$ ,  $0 \leq i \leq n$ . The *coefficient matrix*  $M$  is the matrix of  $\Psi$  in the monomial bases  $B_0, \dots, B_n$ . It is a square matrix and can be divided into four blocks,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

The relation between the coefficient matrix and the multiplication matrix is the following,

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