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Computing individual Kazhdan–Lusztig basis elements ☆

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ABSTRACT

In well-known work, Kazhdan and Lusztig (1979) defined a new set of Hecke algebra basis elements (actually two such sets) associated to elements in any Coxeter group. Often these basis elements are computed by a standard recursive algorithm which, for Coxeter group elements of long length, generally involves computing most basis elements corresponding to Coxeter group elements of smaller length. Thus, many calculations simply compute all basis elements associated to a given length or less, even if the interest is in a specific Kazhdan–Lusztig basis element. Similar remarks apply to "parabolic" versions of these basis elements defined later by Deodhar (1987, 1990), though the lengths involved are the (smaller) lengths of distinguished coset representatives. We give an algorithm which targets any given Kazhdan–Lusztig basis element or parabolic analog and does not precompute any other Kazhdan–Lusztig basis elements. In particular it does not have to store them. This results in a considerable saving in memory usage, enabling new calculations in an important case (for finite and algebraic group 1-cohomology with irreducible coefficients) analyzed by Scott–Xi (2010).

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1. Introduction

This note addresses a need we have perceived for a non-recursive algorithm focused on determining coefficients in Kazhdan–Lusztig polynomials $P_{x,y}$ associated to a single y in a given Coxeter group W, or equivalently, to that of a single Kazhdan–Lusztig Hecke algebra basis element C'_y in the notation of Kazhdan and Lusztig (1979) or Deodhar (1990, p. 101). Our approach here applies also to the parabolic Kazhdan–Lusztig polynomials $P^J_{x,y}$ and basis elements ${}^JC'_y$ (for an appropriate Hecke algebra right module $M = M^J$) in the notation of Deodhar (1990, p. 113). The parabolic notations are defined only for y "distinguished" (shortest) in its right coset $W_J y$ in W, and there is a similar requirement on x.

We follow the notation of Deodhar (1990) closely. The Hecke algebra of W is denoted $\mathcal{H}$. It is a free R-module, where R is the ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$, with basis elements T_x, $x \in W$, as discussed in Deodhar (1990, §3), following standard terminology. The identity element of W is denoted e, and T_e is the identity of the ring $\mathcal{H}$. The set J is a subset of the set S of fundamental generators of W and serves as a set of fundamental generators of the Coxeter group W_J. The set of distinguished right coset representatives of W_J in W is denoted W^J. Henceforth, we fix a subset J, which may be the empty set. The module $M = M^J$ has a basis $\{m_x\}_{x \in W^J}$ with $m_x = m_e T_x$ for $x \in W^J$ and $m_e T_w = q^{\ell(w)} m_e$ for $w \in W_J$. See the displayed action (Deodhar, 1990, p. 113) of $\mathcal{H}$ on M. We mention that the cited display corrects an earlier misprint in the middle term of a similar display (Deodhar, 1987, p. 485). We also remark that the modules considered there and here are "tensor induced" from evident rank 1 modules for the Hecke algebra corresponding to W_J. (Though M is a right $\mathcal{H}$-module, the action of the commutative ring R is often written on the left.) With this terminology, we have

$$ {}^JC'_y = q^{-\ell(y)/2} \sum_{x \le y} P^J_{x,y}(q) m_e T_x \quad (x, y \in W^J). \tag{$*$} $$

We will return to this equation later. It is part of Deodhar (1990, Prop. 5.1(i)), the parabolic analog of Kazhdan and Lusztig (1979, (1.1.c)). If $s \in S$, we have ${}^{\emptyset}C'_s = C'_s = q^{-1/2}(T_e + T_s)$. When the group W_J is finite, with element w^0_J of maximal length, we have $P^J_{x,y} = P_{w^0_J x, w^0_J y}$. See Deodhar (1987, Prop. 3.4), applied through the duality set-up of Deodhar (1991, Rem. 2.6). It is worth noting that, even when W_J is finite, the basic recursion (Deodhar, 1990, Prop. 5.2(iii))[1] for the parabolic Kazhdan–Lusztig polynomials $P^J_{x,y}$ is much more effective than the corresponding non-parabolic ($J = \emptyset$) recursion for computing the polynomials $P_{w^0_J x, w^0_J y}$. We will call (Deodhar, 1990, Prop. 5.2(iii)) the *Deodhar recursion* (to distinguish it from the more elaborate *Deodhar algorithm* we will discuss later). Explicitly, the Deodhar recursion states the following, with ${}^J\mu(z, y)$ denoting the coefficient of $q^{(\ell(y)-\ell(z)-1)/2}$ in $P^J_{z,y}$:

Let $y, ys \in W^J$ with $s \in S$ and $y < ys$. Then $${}^JC'_y C'_s = {}^JC'_{ys} + \sum^{J}_{\substack{z \in W^J \\ zs < z \text{ or } zs \notin W^J}} {}^J\mu(z, y) C'_z.$$

It makes sense also to call the $J = \emptyset$ case, equivalent to Kazhdan and Lusztig (1979, (2.3b) via (1.1.1c)), the *Kazhdan–Lusztig recursion.*

Next, following Deodhar (1990, p. 114), we define, for each finite sequence $\mathbf{s} = (s_1, s_2, \ldots s_k)$ of elements of S whose product $\pi(\mathbf{s}) = s_1 s_2 \cdots s_k$ has length k, the element

$$ {}^JD'_{\mathbf{s}} = m_e C'_{s_1} C'_{s_2} \cdots C'_{s_k}. \tag{${}^JD'_{\mathbf{s}}$} $$

In our algorithm we need to compute a lot of these, but, fortunately for memory requirements, there is no need to store them. Deodhar (1990, Prop. 5.3(i)) gives closed forms for these elements, though

[1] The reader may notice there is a misprint in part (ii) of the same proposition (Deodhar, 1990, Prop. 5.2), where $-f^J$ should simply be f, representing the expression $q^{1/2} + q^{-1/2}$. This is irrelevant to the recursion in part (iii).