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Maximally positive polynomial systems supported on circuits

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ABSTRACT

A real polynomial system with support $\mathcal{W} \subset \mathbb{Z}^n$ is called *maximally positive* if all its complex solutions are positive solutions. A support $\mathcal{W}$ having $n+2$ elements is called a circuit. We previously showed that the number of non-degenerate positive solutions of a system supported on a circuit $\mathcal{W} \subset \mathbb{Z}^n$ is at most $m(\mathcal{W}) + 1$, where $m(\mathcal{W}) \leq n$ is the degeneracy index of $\mathcal{W}$. We prove that if a circuit $\mathcal{W} \subset \mathbb{Z}^n$ supports a maximally positive system with the maximal number $m(\mathcal{W}) + 1$ of non-degenerate positive solutions, then it is unique up to the obvious action of the group of invertible integer affine transformations of $\mathbb{Z}^n$. In the general case, we prove that any maximally positive system supported on a circuit can be obtained from another one having the maximal number of positive solutions by means of some elementary transformations. As a consequence, we get for each n and up to the above action a finite list of circuits $\mathcal{W} \subset \mathbb{Z}^n$ which can support maximally positive polynomial systems. We observe that the coefficients of the primitive affine relation of such circuit have absolute value 1 or 2 and make a conjecture in the general case for supports of maximally positive systems.

0. Introduction and statement of the main results

We consider systems of n polynomial equations in n variables with real coefficients and monomials having integer exponents. The support of such a system is the set of points $a \in \mathbb{Z}^n$ corresponding to monomials $x^a = x_1^{a_1} \cdots x_n^{a_n}$ appearing with a non-zero coefficient. We are only interested in the

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solutions in the complex torus $(\mathbb{C}^*)^n$ and call them simply complex solutions. By Kouchnirenko's theorem (Kouchnirenko, 1975), the number of isolated complex solutions is bounded from above by the normalized volume of the convex-hull Δ of $\mathcal{W}$, which is the usual Euclidean volume of Δ scaled by $n!$. We will always assume that $\mathcal{W}$ is not contained in some hyperplane of $\mathbb{R}^n$, for otherwise this volume would vanish. Kouchnirenko's bound is attained by *non-degenerate* systems. These are systems whose solutions are non-degenerate, that is, at which the differentials of the defining polynomials are linearly independent. Non-degenerate systems are generic within systems with given support.

A solution of a system is called positive if all its coordinates are positive real numbers. A polynomial system is called *maximally positive* if all its complex solutions are positive solutions. For simplicity, we consider here only non-degenerate systems whose number of complex solutions is the normalized volume of Δ (in other words, systems reaching Kouchnirenko's bound). Any sufficiently small perturbation of the coefficient matrix of such a system produces another non-degenerate system with the same support and the same number of non-degenerate complex solutions.

Consider for instance the case $n=1$ of a polynomial in one variable $f(x)=\sum_{i=1}^{s} c_i x^{a_i} \in \mathbb{R}[x^{\pm 1}]$, where $a_i < a_{i+1}$ for $i=1,\dots,s-1$ and all coefficients c_i are non-zero. This polynomial is maximally positive if all its complex roots are positive. It follows from Descartes' rule of signs that if f is maximally positive then $a_{i+1}=a_i+1$ and $c_i \cdot c_{i+1} < 0$ for $i=1,\dots,s-1$.

Another example is provided by systems with support the set of vertices $\mathcal{W}=\{w_0, w_1, \dots, w_n\}$ of an n-dimensional simplex in $\mathbb{R}^n$. Multiplying each equation by x^{-w_0} if necessary, we may assume that w_0 is the origin. Then Gaussian elimination transforms this system into an equivalent system of the form $x^{w_i}=c_i$, $i=1,\dots,n$, where $c_1,\dots,c_n$ real non-zero numbers. The number of complex solutions of this last system is the absolute value of the determinant of the matrix with columns w_i for $i=1,\dots,n$, which is precisely the normalized volume of the convex-hull of $\mathcal{W}$. On the other hand, it is easy to see that such a system has at most one positive solution. It follows that if $\mathcal{W}=\{w_0, w_1, \dots, w_n\}$ is the support of a maximally positive system, then the vectors $w_i - w_0$ for $i=1,\dots,n$ generate the lattice $\mathbb{Z}^n$.

In the general case, it is not difficult to show that if $\mathcal{W} \subset \mathbb{Z}^n$ is the support of a maximally positive polynomial system, then the integer affine span $\mathbb{Z}\mathcal{W}$ of $\mathcal{W}$ is equal to $\mathbb{Z}^n$ (see Proposition 2.2). Such supports are called *primitive*.

In the present paper, we initiate the study of maximally positive systems in the first non-trivial case: when the support of the system is a (possibly degenerate) circuit. We define a circuit in $\mathbb{Z}^n$ as a subset of $n+2$ points. Up to renumbering the elements of a circuit $\mathcal{W}=\{w_1,\dots,w_{n+2}\} \subset \mathbb{Z}^n$, there is only one affine relation

$$\sum_{i=1}^{s} \lambda_i w_i = \sum_{i=s+1}^{n+2} \lambda_i w_i, \tag{1}$$

where the λ_i's are nonnegative coprime integer numbers and $\sum_{i=1}^{s} \lambda_i = \sum_{i=s+1}^{n+2} \lambda_i$. This affine relation is called the *primitive affine relation* of $\mathcal{W}$. The *degeneracy index* $m(\mathcal{W})$ is the dimension of the affine span of a minimal affinely dependent subset. Thus $1 \le m(\mathcal{W}) \le n$ and $\mathcal{W}$ is called *non-degenerate* when $m(\mathcal{W})=n$. Equivalently, $m(\mathcal{W})+2$ is the number of non-zero coefficients in (1). The following result has been proved in Bihan (2007).

Theorem 0.1. *(See (Bihan, 2007).) The number of positive solutions of a polynomial system supported on a circuit $\mathcal{W} \subset \mathbb{Z}^n$ does not exceed $m(\mathcal{W})+1$. Moreover, for any positive integers m, n with $m \le n$, there exists a circuit $\mathcal{W} \subset \mathbb{Z}^n$ such that $m(\mathcal{W})=m$, and a polynomial system with support $\mathcal{W}$ and having $m(\mathcal{W})+1$ positive solutions.*

The proof for the sharpness (the second assertion of Theorem 0.1) given in Bihan (2007) uses the notion of *real dessins d'enfant*. This technique is constructive but explicit values for the coefficients of the system are not given. Explicit systems reaching this bound have been given by Kaitlyn Phillipson and J. Maurice Rojas (2013). It turns out that the polynomial systems constructed in Bihan (2007) are maximally positive: they have $m(\mathcal{W})+1$ positive solutions and no other complex solutions. The corresponding dessins d'enfant have a lot of symmetries, and it was then natural to ask to what extent