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The computation of generalized Ehrhart series in Normaliz

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ABSTRACT

We describe an algorithm for the computation of generalized (or weighted) Ehrhart series based on Stanley decompositions as implemented in the offspring NmzIntegrate of Normaliz. The algorithmic approach includes elementary proofs of the basic results. We illustrate the computations by examples from combinatorial voting theory.

Let $M \subset \mathbb{Z}^n$ be an affine monoid endowed with a positive $\mathbb{Z}$-grading deg. Then the *Ehrhart* or *Hilbert series* is the generating function

$$E_M(t) = \sum_{x \in M} t^{\deg x} = \sum_{k=0}^{\infty} \#\{x \in M\colon \deg x = k\} t^k,$$

and $E(M,k) = \#\{x \in M\colon \deg x = k\}$ is the Ehrhart or Hilbert function of M (see Bruns and Gubeladze, 2009 for terminology and basic theory). It is a classical theorem that $E_M(t)$ is the power series expansion of a rational function of negative degree at $t_0 = 0$ and that $E(M,k)$ is given by a quasipolynomial

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of degree $\operatorname{rank} M - 1$ with constant leading coefficient equal to the (suitably normed) volume of the rational polytope

$$P = \operatorname{cone}(M) \cap A_1$$

where $\operatorname{cone}(M) \subset \mathbb{R}^n$ is the cone generated by M and A_1 is the hyperplane of degree 1 points. See Beck and Robins (2007) for a gentle introduction to the fascinating area of Ehrhart series. In the following we assume that

$$M = \operatorname{cone}(M) \cap L$$

for a sublattice L of $\mathbb{Z}^n$. Then $E(M, k)$ counts the L-points in the multiple kP, and is therefore the Ehrhart function of P (with respect to L).

Monoids of the type just introduced are important for applications, and in some of them, like those discussed in Section 3, one is naturally led to consider *generalized* (or weighted) *Ehrhart series*

$$E_{M,f}(t) = \sum_{x \in M} f(x) t^{\deg x}$$

where f is a polynomial in n indeterminates. It is well-known that also the generalized Ehrhart series is the power series expansion of a rational function; see Baldoni et al. (2011; 2012).

In applications that involve strict linear inequalities M is to be replaced by $M' = M \cap (\operatorname{cone}(M) \setminus \mathscr{F})$ where $\mathscr{F}$ is a union of faces (not necessarily facets) of $\operatorname{cone}(M)$. Our approach covers this "semi-open" situation as well.

In 2012 we have implemented an offspring of Normaliz (Bruns et al., no date) called NmzIntegrate[1] that computes generalized Ehrhart series. The input polynomials of NmzIntegrate must have rational coefficients, and we assume that f is of this type although it is mathematically irrelevant. This note describes the computation of generalized Ehrhart series based on Stanley decompositions (Stanley, 1982). Apart from taking the existence of Stanley decompositions as granted, we give complete and very elementary proofs of the basic facts. They follow exactly the implementation in NmzIntegrate (or vice versa). The semi-open case mentioned above has already been implemented in the current development versions of Normaliz and NmzIntegrate. It will be contained in the next public version.

The generalized Ehrhart function is given by a quasipolynomial $q(k)$ of degree $\le \deg f + \operatorname{rank} M - 1$, and the coefficient of $k^{\deg f + \operatorname{rank} M - 1}$ in $q(k)$ can easily be described as the integral of the highest homogeneous component of f over the polytope P. Therefore we have also included (and implemented) an approach to the computation of integrals of polynomials over rational polytopes in the spirit of the Ehrhart series computation. See Baldoni et al. (2012) and De Loera et al. (2013) for more sophisticated approaches. Our algorithm and its implementation in NmzIntegrate have been developed independently from LattE integrale (DeLoera et al., no date). It is a consequent extension of the Normaliz algorithm for the computation of ordinary Ehrhart series.

1. The computation of generalized Ehrhart series

Via a Stanley decomposition and substitution the computation of generalized Ehrhart series can be reduced to the case in which M is a free monoid, and for free monoids one can split off the variables of f successively so that one ends at the classical case $M = \mathbb{Z}_+$. We take the opposite direction, starting from $\mathbb{Z}_+$.

1.1. The monoid $\mathbb{Z}_+$

Let $M = \mathbb{Z}_+$. By linearity it is enough to consider the polynomials $f(k) = k^m$, $k \in \mathbb{Z}_+$, for which the generalized Ehrhart series is given by

[1] NmzIntegrate version 1.2 is available as part of the Normaliz 2.11 distribution.