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Explicit computations of invariants of plane quartic curves

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ABSTRACT

We establish a complete set of invariants for ternary quartic forms. Further, we express four classical invariants in terms of these generators.

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1. Introduction

Studying rings of invariants is classical (19th century) algebra. For example, it was known in those days that a ternary cubic has two invariants S and T of degrees 4 and 6, respectively. *Every invariant of the cubic can be expressed as a rational function of S and T* (Salmon, 1873, p. 186). Here, *rational* should be interpreted as being a polynomial.

The ring of invariants of ternary quartics is more complicated. First, one can show that the degree of each invariant is divisible by 3. Shioda (1967) conjectured that the ring of invariants of ternary quartic forms is generated by 13 invariants of degrees 3, 6, 9, 9, 12, 12, 15, 15, 18, 18, 21, 21, 27.

Dixmier (1987) proved that invariants of degree 3, 6, 9, 12, 15, 18, 27 form a complete system of primary invariants (Derksen and Kemper, 2002, Def. 2.4.6). Further, he proved that at most 56 invariants suffice to generate the entire ring of invariants.

Using the Clebsch transfer principle and invariants of binary quartics, we describe an efficient algorithm to compute invariants of the degrees given above. As an application, we show that Shioda was right. Finally, we express four classical invariants in terms of the listed generators.

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2. Invariants, covariants, and contravariants

The standard left action of $\mathrm{Gl}_n(\mathbb{C})$ on $\mathbb{C}^n$ induces a right action on the polynomial ring $\mathbb{C}[X_1,\dots,X_n]$ by $f^M := f(MX)$.

Let $\mathbb{C}[X_1,\dots,X_n]_d$ be the space of homogeneous polynomials of degree d. A polynomial mapping $I\colon \mathbb{C}[X_1,\dots,X_n]_d \to \mathbb{C}$ is called an *invariant* if

$$I\big(f^M\big) = I(f)\det(M)^e$$

holds for some $e \in \mathbb{Z}$ and all $M \in \mathrm{Gl}_n(\mathbb{C})$, $f \in \mathbb{C}[X_1,\dots,X_n]_d$. The *degree* of I is the degree of I as a polynomial in the coefficients of f.

A *covariant* c is a mapping $c\colon \mathbb{C}[X_1,\dots,X_n]_{d_1} \to \mathbb{C}[X_1,\dots,X_n]_{d_2}$ with

$$c\big(f^M\big) = c(f)^M \det(M)^e$$

for some $e \in \mathbb{Z}$ and all $M \in \mathrm{Gl}_n(\mathbb{C})$, $f \in \mathbb{C}[X_1,\dots,X_n]_{d_1}$.

A *contravariant* C is a mapping $C\colon \mathbb{C}[X_1,\dots,X_n]_{d_1} \to \mathbb{C}[X_1,\dots,X_n]_{d_2}$ with

$$C\big(f^M\big) = C(f)^{(M^\top)^{-1}} \det(M)^e$$

for some $e \in \mathbb{Z}$ and all $M \in \mathrm{Gl}_n(\mathbb{C})$, $f \in \mathbb{C}[X_1,\dots,X_n]_{d_1}$.

We call d_2 the *order* of the covariant (resp. contravariant). The *degree* of a covariant (resp. contravariant) is the degree of its coefficients viewed as polynomials in the coefficients of f.

Some examples are as follows:

- An invariant of a quadratic form in n variables is given by the determinant of its matrix. It is of degree n.
- More generally, the discriminant of a form of degree d in n variables is an invariant of degree $n(d-1)^{n-1}$.
- A covariant of a form is given by its Hessian $\det(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j}$. It has degree n and order $n(\deg(f)-2)$.

Remarks.

- Invariants are called invariants because they are invariant with respect to the action of $\mathrm{Sl}_n(\mathbb{C})$.
- The sets of all invariants, covariants, or contravariants are rings.
- The ring of covariants (resp. contravariants) is a module over the ring of invariants.
- By a theorem of Hilbert, the ring of all invariants is finitely generated.

3. Invariants for binary quartics

A classical way to write down invariants is the symbolic form. Nowadays, this seems to be almost completely forgotten. We refer to Salmon (1885, 1873) for a detailed explanation and Weyl (1973, Chap. VIII, Sec. 2) for a more recent treatment. A brief summary is given in Hunt (1996, App. B).

Write a general binary quartic in the form $f(x,y) := ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4$. In degrees 2 and 3, it has the invariants $S_2 := 96(12ae - 3bd + c^2)$ and $T_2 := 192(72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3)$.

In symbolic notation, these invariants are given by $(1\,2)^4$ and $(1\,2)^2(1\,3)^2(2\,3)^2$. Here, $(i\,j)$ is an abbreviation for the differential operator

$$\det\begin{pmatrix} \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial y_j} \end{pmatrix}.$$

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