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Systems of equations with a single solution

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ABSTRACT

We classify generic systems of polynomial equations with a single solution, or, equivalently, collections of lattice polytopes of minimal positive mixed volume. As a byproduct, this classification provides an algorithm to evaluate the single solution of such a system.

1. Introduction

The mixed volume is the unique symmetric real-valued function MV of n convex bodies in an n-dimensional vector space V such that

$$\mathrm{MV}(A+B, A_2, \dots, A_n) = \mathrm{MV}(A, A_2, \dots, A_n) + \mathrm{MV}(B, A_2, \dots, A_n)$$

in the sense of Minkowski summation $A+B = \{a+b \mid a \in A,\ \ b \in B\}$, and $\mathrm{MV}(A, \dots, A) =$ (volume of A) in the sense of a given volume form on V. In what follows, V is always of the form $\mathcal{V} \otimes \mathbb{R}$, where $\mathcal{V}$ is an integer lattice, and the volume form is always chosen in such a way that the volume of the torus $V/\mathcal{V}$ is equal to $n!$. For this volume form, the mixed volume of lattice polytopes (i.e. the ones whose vertices are in $\mathcal{V}$) is always integer.

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The mixed volume is always non-negative. A criterion for a collection of polytopes to have zero mixed volume was established by Minkowski (1911) in the original paper, where multidimensional mixed volumes were introduced: the mixed volume of bodies is positive if and only if they contain linearly independent segments. A constructive version of this criterion was given by D. Bernstein and A. Khovanskii (Khovanskii, 1978, see, e.g., Esterov (2010, Lemma 1.2) for a proof):

Proposition 1. *The mixed volume of convex bodies is zero if and only if k of these bodies sum up to a body of dimension strictly smaller than k.*

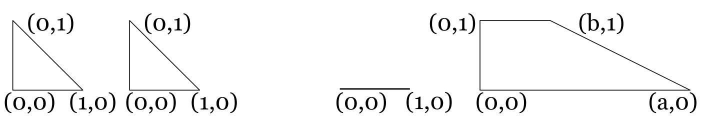
In this paper, we provide the classification of collections of lattice polytopes with the minimal positive mixed volume (by induction on n):

Theorem 1. *A collection A of n lattice polytopes in V has the unit mixed volume if and only if*

1) *the mixed volume is not zero, and*
2) *there exists $k > 0$ such that, up to translations, k of the polytopes are faces of the same k-dimensional volume 1 lattice simplex in a k-dimensional rational subspace $U \subset V$, and the images of the other $n-k$ polytopes under the projection $V \to V/U$ have the unit mixed volume.*

See Section 2 for the proof. Here and in what follows, the volume forms in the subspace $U \subset V$ and in the quotient space V/U are induced by the lattices $\mathcal{U} = U \cap \mathcal{V}$ and $\mathcal{V}/\mathcal{U}$ respectively. Condition (1) implies that the k polytopes in (2) generate the whole simplex.

Example. Any pair of lattice polygons of mixed area 1 is equal (up to translations and an authomorphism of $\mathbb{Z}^2$) to exactly one of the following pairs (with $a \geqslant b \geqslant 0$):



The question of classifying lattice polytopes of small mixed volume is particularly motivated by the study of codimensions of discriminants (see, e.g., Esterov, 2009, Theorem 3.13 or Cattani et al., 2013, for details). Theorem 1 was conjectured in Esterov (2009, Conjecture 3.16) and its special case of full-dimensional polytopes was proved in Cattani et al. (2013, Proposition 2.7).

The mixed volume is related to algebra by the Kouchnirenko–Bernstein formula: a system of n polynomial equations of n variables with Newton polytopes $N_1, N_2, \dots, N_n$ and generic coefficients has $\mathrm{MV}(N_1, N_2, \dots, N_n)$ solutions in the complex torus $(\mathbb{C}\setminus 0)^n$, see Bernstein (1975). Thus, Theorem 1 classifies all generic systems of polynomial equations with a unique solution. By a general Gröbner basis or Galois theory argument, the solution of such a system admits a rational expression in terms of the coefficients of the system, and Theorem 1 provides an explicit construction for it (by induction on n):

I) upon a certain monomial change of variables, k of the n equations become linear (non-homogeneous) equations of k variables, from which the k variables can be evaluated;
II) after the substitution of the evaluated variables in the other equations, we obtain $n-k$ generic equations of $n-k$ variables with a unique solution and proceed to the next group of simultaneously linearizable equations.

Example. In order to solve a system $a + bxy = 0$, $f(xy) + yg(xy) = 0$, we first make a monomial change $xy = u$, $y = v$, then solve the first equation $a + bu = 0$, linear in u, and obtain $u = -a/b$, then put the result to the second equation $f(-a/b) + vg(-a/b) = 0$, linear in v, and obtain