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Separating inequalities for nonnegative polynomials that are not sums of squares

Sadik Iliman [a], Timo de Wolff [b]

[a] Goethe-Universität, FB 12 – Institut für Mathematik, Postfach 11 19 32, 60054 Frankfurt am Main, Germany
[b] Universität des Saarlandes, Fachrichtung Mathematik, Postfach 15 11 50, 66041 Saarbrücken, Germany

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ABSTRACT

Ternary sextics and quaternary quartics are the smallest cases where there exist nonnegative polynomials that are not sums of squares (SOS). A complete classification of the difference between these cones was given by G. Blekherman via analyzing the extreme rays of the corresponding dual cones. However, an exact computational approach in order to build separating extreme rays for nonnegative polynomials that are not sums of squares is a widely open problem. We provide a method substantially simplifying this computation for certain classes of polynomials on the boundary of the PSD cones. In particular, our method yields separating extreme rays for every nonnegative ternary sextic with at least seven zeros, which proves a slight variation of a conjecture by Blekherman for many instances. As an application, we compute rational certificates for some prominent polynomials.

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1. Introduction

We consider real polynomials in the vector space of all homogeneous polynomials in n variables of degree d, denoted by $H_{n,d}$. For every $p \in H_{n,d}$ we denote its real projective variety as $\mathcal{V}(p)$. Let $P_{n,d} \subset H_{n,d}$ be the cone of all nonnegative polynomials in n variables of degree d.

Inside $H_{n,2d}$, there are two full dimensional convex cones of special interest, the cone of nonnegative polynomials and the cone of sums of squares (for a general background about nonnegative

E-mail addresses: iliman@math.uni-frankfurt.de (S. Iliman), dewolff@math.uni-sb.de (T. de Wolff).

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polynomials and sums of squares see e.g. Delzell and Prestel, 2001; Lasserre, 2000/2001; Marshall, 2008; Reznick, 2000; for some metric and convexity properties of these cones see Blekherman, 2004).

$$P_{n,2d} := \left\{ f \in H_{n,2d}\colon\ f(x) \geq 0,\ \text{for all } x \in \mathbb{R}^n \right\},$$
$$\Sigma_{n,2d} := \left\{ f \in P_{n,2d}\colon\ f = \sum_i f_i^2 \text{ for some } f_i \in H_{n,d} \right\}.$$

The investigation of the relationship between the cone of nonnegative polynomials and the cone of sums of squares began in the seminal work of Hilbert when he showed that the cone of nonnegative polynomials coincides with the cone of sums of squares exactly in the cases of bivariate forms ($n = 2$), quadratic forms ($2d = 2$) and ternary quartics ($n = 3, 2d = 4$) (Hilbert, 1888).

The Motzkin polynomial $m(x, y, z) := x^4y^2 + x^2y^4 - 3x^2y^2z^2 + z^6$ was the first explicitly known example for a nonnegative polynomial which is not a sum of squares. Most proofs for this fact are based on term by term inspections (see e.g. Motzkin, 1983; Reznick, 2000). In near past other proofs were found, e.g. using representation theory (see Bosse, 2007).

In Blekherman (2006), Blekherman showed that for fixed dimension $2d \geq 4$ there are significantly more nonnegative polynomials than sums of squares as n tends to infinity. However, the question of precisely when nonnegative polynomials begin to significantly overtake sums of squares is much less understood. In the smallest cases where there exist nonnegative polynomials which are not sums of squares ($(n, 2d) = (3, 6), (4, 4)$) the general conjecture is that these two cones do not differ very much. This conjecture is supported by the following two facts: Firstly, the maximal dimensional difference between exposed faces of the cone of nonnegative polynomials and sums of squares is one (see Blekherman et al., 2013). Secondly, all extreme rays of the dual sums of squares cone $\Sigma_{3,6}^*$ have rank one or rank seven (see Blekherman, 2012).

Recently, in Blekherman et al. (2012), it is shown that, except the discriminant, there is a unique component of the algebraic boundary of $\Sigma_{3,6}$ with degree 83 200, which indicates the complicated structure of the SOS cone. But still, the geometry and the relationship between these two cones in the smallest cases are less understood.

In the smallest cases $(n, 2d) = (3, 6)$ and $(n, 2d) = (4, 4)$, Blekherman shows that it is precisely the Cayley–Bacharach relation that prevents sums of squares from filling out the cone of nonnegative polynomials. More precisely, in Blekherman (2012), it is shown that every separating extreme ray in the dual SOS cone for a given nonnegative polynomial that is not a sum of squares depends on a 9-point configuration for $(n, 2d) = (3, 6)$ resp. an 8-point configuration for $(n, 2d) = (4, 4)$ coming from the intersection of two qubic resp. three quadric polynomials. Furthermore, given an appropriate 9-point (resp. 8-point) configuration, one can write down an extreme ray of the dual SOS cone (see Theorems 2.6 and 2.7) corresponding to faces of maximal dimension of the SOS cone. In Blekherman (in press), Blekherman extends the investigation of the extreme rays of the dual sums of squares cones, especially for ternary forms.

A central problem in this area is how to determine the separating inequalities efficiently. This can always be done in a numerical way (see Section 2.2), but is widely open for exact methods currently. Of course, symbolic results are strongly preferred – not only since they provide algebraic certificates, which are exact. Furthermore, they can also be connected to the whole machinery of algebraic geometry and thus be used to tackle follow-up questions (like the semi-algebraic description of appropriate nine point configurations for the Motzkin polynomial, which we provide here; see e.g., Fig. 1). Hence, finding constructive symbolic methods for computing these inequalities is one main research issue. Blekherman's result does not provide an efficient symbolic way to obtain a proper 9-point (resp. 8-point) configuration to solve this problem (see Section 2.2 for further details).

The key idea of this article is to construct a proper 9-point (resp. 8-point) configuration out of a given initial set of points. Specifically, we investigate nonnegative polynomials p which lie on the boundary of the cones $P_{3,6}$ and $P_{4,4}$ (which cover most of the explicitly known nonnegative polynomials that are not SOS, see Blekherman, 2006). Our main result, Theorem 3.1, provides a sufficient condition for using k zeros of p as a subset of a 9-point (resp. 8-point) configuration. The idea is to fill up the set of k zeros with $9-k$ (resp. $8-k$) points such that a genericity and a quadratic condition