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On Alexander–Conway polynomials of two-bridge links

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ABSTRACT

We consider Conway polynomials of two-bridge links as Euler continuant polynomials. As a consequence, we obtain new and elementary proofs of classical Murasugi's 1958 alternating theorem and Hartley's 1979 trapezoidal theorem. We give a modulo 2 congruence for links, which implies the classical Murasugi's 1971 congruence for knots. We also give sharp bounds for the coefficients of Euler continuants and deduce bounds for the Alexander polynomials of two-bridge links. These bounds improve and generalize those of Nakanishi–Suketa's 1996. We easily obtain some bounds for the roots of the Alexander polynomials of two-bridge links. This is a partial answer to Hoste's conjecture on the roots of Alexander polynomials of alternating knots.

1. Introduction

In this paper, we consider the Conway polynomial of a two-bridge link as an Euler continuant polynomial. We study the problem of determining whether a given polynomial is the Conway polynomial of a two-bridge link (or knot), or equivalently, if it is an Euler continuant polynomial. For small degrees, this problem can be solved by an exhaustive search of possible two-bridge links. Here, we give necessary conditions on the coefficients of the polynomial, which can be tested for high degree polynomials.

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In Section 2 we present Euler continuant polynomials and give some properties of their coefficients. We show their relations with the Fibonacci polynomials f_k defined by:

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+2}(z) = z f_{n+1}(z) + f_n(z).$$

In Section 3, we recall the definitions of two-bridge links and we present the description of the Conway polynomial of a two-bridge link as an extended Euler continuant polynomial. We obtain a characterization of modulo 2 two-bridged Conway polynomials.

Theorem 3.5. *Let $\nabla(z) \in \mathbf{Z}[z]$ be the Conway polynomial of a two-bridge link (or knot). There exists a Fibonacci polynomial $f_D(z)$ such that $\nabla(z) \equiv f_D(z) \pmod 2$.*

We give a simple method (Algorithm 3.6) that determines the integer D such that $\nabla(z) \equiv f_D(z) \pmod 2$. This is used to test when $\nabla(z) \equiv 1 \pmod 2$, which is a necessary condition to be a two-bridge Lissajous knot.

In Section 4, we find inequalities for the coefficients of the Conway polynomials of two-bridge links denoted by

$$\nabla_m(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k} z^{m-2k}.$$

Theorem 4.1. *For $k \geq 0$,*

$$|c_{m-2k}| \leq \binom{m-k}{k} |c_m|.$$

If equality holds for some integer $0 < k < \lfloor \frac{m}{2} \rfloor$, then it holds for all integers $0 \leq k \leq \lfloor \frac{m}{2} \rfloor$. In this case, the link is isotopic to the link $C(2, 2, \ldots, 2)$, or to the torus link $\mathrm{T}(2, m) = C(2, -2, \ldots, (-1)^{m-1}2)$, up to mirror symmetry.

When $|c_m| \neq 1$, we have the following sharper bounds:

Theorem 4.4. *Let $g \geq 1$ be the greatest prime divisor of c_m, and let $k \neq 0$. Then*

$$|c_{m-2k}| \leq \left(\binom{m-k-1}{k} + \frac{1}{g} \left(\binom{m-k-1}{k-1} - 1 \right) \right) |c_m| + 1.$$

Equality holds for $C(2g, 2, \ldots, 2)$ and $C(2g, -2, 2, \ldots, (-1)^{m-1}2)$.

In Section 5, we apply our results to the Alexander polynomials. Our modulo 2 congruence of Theorem 3.5 provides a simple proof of a congruence of Murasugi (1971) for periodic knots (two-bridge knots have period two). Moreover, we deduce a congruence for the Hosokawa polynomials of two-bridge links (Corollary 5.5). Then, we obtain a simple proof of both the Murasugi alternating theorem (Murasugi, 1958, 1996) and the Hartley trapezoidal theorem (Hartley, 1979) (see also Kanenobu, 1984) using the trapezoidal property:

Theorem 4.6. *Let K be a two-bridge link (or knot). Let*

$$\nabla_K = c_m \left(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m-2i+1} \right), \quad \alpha_0 = 1$$

be its Conway polynomial written in the Fibonacci basis. Then we have

1. $\alpha_j \geq 0$, $j = 0, \ldots, \lfloor \frac{m}{2} \rfloor$.
2. *If $\alpha_i = 0$ for some $i > 0$ then $\alpha_j = 0$ for $j \geq i$.*