Journal of Symbolic Computation 68 (2015) 297–307

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Combinatorial excess intersection

Jose Israel Rodriguez

Department of Applied and Computational Mathematics and Statistics, 153 Hurley Hall, Notre Dame, IN 46556, United States

ARTICLE INFO

Article history:
Received 15 October 2013
Accepted 29 March 2014
Available online 28 September 2014

Keywords:
Excess intersection
Newton polytopes
Combinatorics
Numerical algebraic geometry
Mixed volume

ABSTRACT

We provide formulas and develop algorithms for computing the excess numbers of an ideal. The solution for monomial ideals is given by the mixed volumes of polytopes. These results enable us to design numerical algebraic geometry homotopies to compute excess numbers of any ideal.

1. Introduction

Consider a homogeneous ideal $\mathcal{I} \subset \mathbb{C}[x_0, \ldots, x_n]$, and let $f_1, \ldots, f_n$ be homogeneous polynomials in $\mathcal{I}$. Since $(f_1, \ldots, f_n) \subset \mathcal{I}$, we have $\mathbf{V}(f_1, \ldots, f_n) \supset \mathbf{V}(\mathcal{I})$. The *excess intersection* of the variety of $(f_1, f_2, \ldots, f_n)$ with respect to the variety of $\mathcal{I}$ is defined as the quasiprojective variety $\mathbf{V}(f_1, \ldots, f_n) \backslash \mathbf{V}(\mathcal{I})$. We define the *excess number* $E_\bullet(\mathcal{I}; f_1, \ldots, f_n)$ of an ideal $\mathcal{I}$ to be the number of solutions in $\mathbf{V}(f_1, \ldots, f_n) \backslash \mathbf{V}(\mathcal{I})$.

Excess intersections are a well studied problem with applications in enumerative geometry, machine learning Király et al. (2012a, 2012b), and algebraic statistics (Jost, 2013). In addition, there is a well developed theory of Segre classes to study this problem that has been exploited in Bates et al. (2013), Di Rocco et al. (2011), Moe and Qviller (2012) using computational algebraic geometry as well. Recent work by Paolo Aluffi has pushed this area even further in Aluffi (2013). However, the motivation for this paper came at the 2012 Institute for Mathematics and its Applications Participating Institution Summer Program for Graduate Students in Algebraic Geometry for Applications by

E-mail address: jo.ro@berkeley.edu.

http://dx.doi.org/10.1016/j.jsc.2014.09.034
0747-7171/

Mike Stillman. We will focus on the numerical algebraic geometry perspective, where it is ideal to solve *square* systems of equations, meaning the number of unknowns equals the number of equations. So by understanding the zero-dimensional solutions of an excess intersection of an ideal, we can study the ideal itself. Our computations were performed with `Bertini` (Bates et al.), `PHCpack` (Verschelde), and `Macaulay2` (Grayson and Stillman).

We begin our study in the case that $\mathcal{I}$ is an ideal generated by $B_1, B_2, \ldots, B_l$, and $f_1, \ldots, f_n$ define a $B_{\mathcal{I}}$-system of equations of degree $(d_1, d_2, \ldots, d_n)$.

Definition 1. Let $\mathcal{I}$ be an ideal of $\mathbb{C}[x_0, \ldots, x_n]$ generated by $B_1, \ldots, B_l$ whose respective degrees are $p_1, \ldots, p_l$. Suppose $(d_1, d_2, \ldots, d_n)$ is such that

$$\min(d_1, d_2, \ldots, d_n) \geq \max(p_1, \ldots, p_l).$$

Let a_{ij} denote a form of degree $d_i - \deg B_j$. If the forms $f_1, \ldots, f_n$ are given by

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1l} \\ a_{21} & \cdots & a_{2l} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nl} \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_l \end{bmatrix},$$

then we say $f_1, \ldots, f_n$ are a $B_{\mathcal{I}}$-system of degree $(d_1, d_2, \ldots, d_n)$.

The space of $B_{\mathcal{I}}$-system's with degree $(d_1, d_2, \ldots, d_n)$ is parameterized by the coefficients of the homogeneous polynomials a_{ij}. If $\mathcal{I}$ is generated by $B_1, \ldots, B_l$, then we denote the excess number of a general $B_{\mathcal{I}}$-system with degree $(d_1, d_2, \ldots, d_n)$ as $E_{\bullet}(\mathcal{I}; d_1, \ldots, d_n)$.

In the first section we will be interested in determining excess numbers of $B_{\mathcal{I}}$-system's where $B_1, \ldots, B_l$ are monomials.

At times it will be more convenient to work with the equivalence number

$$E_{\circ}(\mathcal{I}; d_1, \ldots, d_n) := d_1 \cdots d_n - E_{\bullet}(\mathcal{I}; d_1, \ldots, d_n).$$

This definition is inspired by the notion of the *equivalence* in Fulton (1998), Chapter 6. This number is the difference between the Bezout bound and the excess number in the cases we consider. The contributions of the paper include numerical algebraic geometry algorithms to compute excess numbers and a combinatorial proof of the theorem below. This theorem can be proven easily using Fulton–MacPherson intersection theory, and in fact doing so generalizes the result to any ideal generated by a regular sequence. But in the proof we present, we will see how $E_{\bullet}(\mathcal{I}; d_1, \ldots, d_n)$ and $E_{\circ}(\mathcal{I}; d_1, \ldots, d_n)$ relate to the volume of a subdivided simplex. The algorithms we present take advantage of the polyhedral structure in our problem to give bounds (lower and upper-bound) for an excess number.

Theorem 2. *Let $\mathcal{I}$ be an ideal of $\mathbb{C}[x_0, \ldots, x_n]$ generated by $B_1, B_2, \ldots, B_k$ such that $B_i = x_i^{p_i}$. If $f_1, \ldots, f_n$ define a $B_{\mathcal{I}}$-system of degree $(d_1, d_2, \ldots, d_n)$, then*

$$E_{\bullet}(\mathcal{I}; d_1, \ldots, d_n) + p_1 \cdots p_k \sum_{\delta=0}^{n-k} \left((-1)^{\delta} \mathcal{D}_{n-k-\delta} \mathcal{P}_{\delta}\right) = d_1 d_2 \cdots d_n$$

where $\mathcal{D}_{n-k-\delta}$ is the degree $n-k-\delta$ elementary symmetric function evaluated at $d_1, \ldots, d_n$ and P_{δ} is the degree δ complete homogeneous symmetric function evaluated at $p_1, \ldots, p_k$.

The paper is structured as follows. We consider the case when $\mathcal{I}$ is a monomial ideal, and show excess numbers equal mixed volumes of polytopes (Lemma 7). By further restricting to the case when the ideal $\mathcal{I}$ defines a complete intersection that is also a linear space (though not necessarily reduced), we do a mixed volume computation (Lemma 10) to get an explicit formula for excess numbers. In the final section, we present our algorithms that take advantage of the first sections results.