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A procedure for computing the symmetric difference of regions defined by polygonal curves ☆

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ABSTRACT

Given any two regions A, B in the plane, defined by polygonal (simple, closed and oriented) curves, associated with their respective boundaries, we describe a procedure to compute the symmetric difference $A \oplus B$. The output is also presented in the form of polygonal curves, where in particular the curves describing the union $A \cup B$, the intersection $A \cap B$, the difference $A \setminus B$, and the complement of the difference $B \setminus A$, are also obtained. This is related with the two equivalent formulas to compute the symmetric difference, namely $A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

1. Introduction

In this note we provide a detailed description of a procedure to be used in calculating the symmetric difference for arbitrary regions in the plane. In fact, since the whole process is more topological than geometrical, it can be used to calculate the symmetric difference of any two regions embedded in a oriented 2-manifold, while opening the way for a future study in higher dimensions.

By a region in the plane we mean the result of taking the topological closure of the interior of an arbitrary subset of the plane, more specifically, A is a region if

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$$A = cl\big(int(X)\big)$$

for some $X \in \mathbb{R}^2$.

It is clear that every such region is completely determined by its boundary, together with a specified orientation, used to define the interior of the region. We will assume that if *walking* throughout the oriented boundary, in the positive direction, the interior is always on the left.

The boundary of a region can be determined by a family of simple closed and oriented curves in the plane. For practical reasons it is convenient to consider polygonal curves so that each component is determined by a finite sequence of vertices (Har-Peled and Smorodinsky, 2005; Kedem et al., 1986; Sheng and Meier, 1995; Franklin, 1987; Brewer and Mark, 1986; Gardan and Perrin, 1996).

One of the novelties of this work is that instead of a family of polygonal curves, each one specified by a finite sequence of vertices, we will use a graph

$$E \underset{c}{\overset{d}{\rightrightarrows}} V$$

with a symmetry

$$\varphi : E \to E.$$

A graph is simply a fourth-tuple (E, V, d, c) where E and V are sets, with the interpretation that E is the set of edges and V the set of vertices, together with two maps

$$d, c : E \to V$$

designated by domain map and codomain map, associating a starting point, $d(x) \in V$, and an endpoint, $c(x) \in V$, to each edge $x \in E$, as displayed

$$d(x) \xrightarrow{x} c(x).$$

A symmetry φ on a graph (E, V, d, c) is simply a bijection $\varphi : E \to E$ with $d\varphi = c$, that is, for every $x \in E$, we have

$$d(x) \xrightarrow{x} c(x) = d\big(\varphi(x)\big) \xrightarrow{\varphi(x)} c\big(\varphi(x)\big).$$

It is also clear that if the set of vertices is embedded in the plane then the whole graph is embedded and the result is a oriented curve. In general, if no further restrictions are imposed on the graph, it is not guaranteed that the embedded curve represents the boundary of a region (for example it could happen that the curve is not simple). Nevertheless, there is a simple procedure that can be performed on the graph in such a way that the resulting curve is the boundary of a region (it suffices to invert the orientation of some of the edges in the graph), but this will not be discussed in here because the algorithm that we are presenting does not dependent on this issue.

The main construction of this article is the following one. Given two graphs, G_A and G_B, each one with a symmetry and embedded in the plane (or in any other oriented 2-manifold), we provide a way to construct a new graph $G_{A,B}$ with a symmetry and an induced embedding in the plane, in such a way that: if G_A and G_B represent the boundary of two regions A and B in the plane, then the graph $G_{A,B}$ represents the region $A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$. In particular, we also provide a way (see Section 3) to separate the connected components of $G_{A,B}$ into two parts, one describing the union $A \cup B$, the other describing the intersection $A \cap B$. Moreover, applying the same procedure to the region A and the complement of B (this is done by simply inverting the edges in G_B) we obtain the components of $A \setminus B$ and the complement of $B \setminus A$.

The problem of computing boolean set operations for regions in the plane, while being of great importance with obvious applications to many fields (see for instance Requicha and Voelcker (1985), Carlbom (1987), Mantyla (1986), or more recently Martinez et al. (2009), Peng et al. (2005)) has never been considered, to our knowledge, at this level of generality where such a simple (but non-trivial) solution is possible.