Journal of Symbolic Computation 61–62 (2014) 116–127

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Using Fourier series to analyse mass imperfections in vibratory gyroscopes

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ARTICLE INFO

Article history:
Received 14 April 2013
Accepted 18 June 2013
Available online 21 October 2013

Keywords:
Bryan's effect
Fourier series
Lagrange's equations
Frequency splitting
Capture effect
Mathematica

ABSTRACT

When a vibrating structure is subjected to a rotation, the vibrating pattern rotates at a rate (called the *precession rate*) proportional to the inertial angular rate. This is known as *Bryan's effect* and it is employed to calibrate the resonator gyroscopes that are used to navigate in outer space, the stratosphere and under the polar cap. We study Bryan's effect for a non-ideal resonator gyroscope, using the computer algebra system (CAS) Mathematica to do the analysis involved, rendering this work accessible to undergraduate students with a working knowledge of college calculus and basic physics or mechanics (such as senior Engineering Mathematics students). In this paper the density of a slowly rotating vibrating annular disc is assumed to have small variations circumferentially, enabling a Fourier series representation of the density function. Using a CAS, the Lagrangian of the system of vibrating particles in the disc is calculated and, employing the CAS on the Euler–Lagrange equations, the equations of motion of the vibrating, rotating system are calculated in terms of "fast" variables, enabling us to demonstrate that the mass anisotropy induces a frequency splitting (beats). Unfortunately the fast variables are difficult to analyse (even with the aid of a CAS) and consequently a transformation from fast to slow variables is achieved. These slow variables are the *principal* and *quadrature* vibration amplitudes, *precession rate* and a *phase angle*. The transformation yields a system of four nonlinear ordinary differential equations (ODEs). This system of ODE demonstrates that the Fourier coefficients of the density function influence the precession rate and consequently a gyroscope manufactured from such a disc cannot use Bryan's effect for calibration purposes. Indeed, the CAS visualises that a capture effect occurs with the precession angle that appears to vary periodically and not increase linearly (Bryan's effect) as it would for a perfect structure. Keeping in mind that manufacturing imperfections will always be present in the

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0747-7171/$ – see front matter
http://dx.doi.org/10.1016/j.jsc.2013.10.015

real-world, the analysis shows how such density variations may be minimised. Using a symbolic manipulator such as MATHEMATICA to do the "book-keeping" eliminates the plethora of technical detail that arises during calculations of a highly technical nature. This allows the aforementioned students to focus on the salient parts of the analysis, producing results that might have been beyond their capabilities without the aid of a CAS.

1. Introduction

Real-life examples in applied mathematics are abundant, but, in our opinion, intriguing ones are rare. An applied mathematics undergraduate student with a knowledge of mechanics or physics (such as a senior Engineering Mathematics student) should find an instrument that is used to navigate space, underwater and strato aircraft intriguing. The resonator (or vibratory) gyroscope (RG) is such an instrument and being able to understand and model some of the characteristics of such an instrument using standard calculus and a computer algebra system (CAS) is a pedagogical gem. Even though the mathematics involved in the analysis below is standard undergraduate calculus, the technical complexity involved in some of the calculations might be daunting and even prohibitive if attempted by hand. We demonstrate below how a CAS may be fruitfully utilised to do the "mathematical book-keeping", freeing the student to observe the salient parts of the analysis — they begin to see the wood from the trees. The same may be said about us. Indeed, we do not limit the use of CAS to pedagogical issues alone. We (and various of our co-authors) have utilised CAS in the production and understanding of our own research. This has resulted in, among others, the papers Joubert et al. (2007), Joubert et al. (2009), Joubert et al. (2010) and Shatalov et al. (2011). We are not alone in exploiting CAS for this purpose. Indeed, the authors Abouzahra and Pavelle (1990), Caviness (1986), Eisenberger (1990) and Moses (2012), to mention but a few, have also utilised CAS for teaching and/or enhancing their research.

David Rozelle of the Northrop Grumman Co, Navigation Systems Division, in Woodland Hills, California, USA, states the following: "Small size, low noise, high performance and no wear-out has made the Hemispherical Resonator Gyroscope (HRG) the choice for high value space missions. After 14 years of production the HRG boasts over 12-million operating gyrohours in space with 100% mission success" (Rozelle, 2009).

When a vibrating structure is subjected to an inertial rotation, the vibrating pattern rotates at a rate proportional to the inertial angular rate. This effect, known as "Bryan's effect" in the sequel, was first observed by Bryan (1890). For the constant of proportionality η, Bryan made the following calculation for a body consisting of a ring or cylinder:

$$\eta = \frac{\text{Angular rate of the vibrating pattern}}{\text{Inertial angular rate of the vibrating body}}. \tag{1}$$

This constant of proportionality η has come to be known as "Bryan's factor" and an easy technique for determining this factor is described in Joubert et al. (2010).

A patent for an RG based on Bryan's effect was registered by Loper and Lynch (1990) and production was initiated. However, an RG cannot be manufactured without imperfections (anisotropies), and these imperfections cannot be ignored because they cause departures from ideal mass, stiffness and damping distributions and therefore affect resonator dynamics. The equations of motion of an RG consisting of a slowly rotating solid cylindrical disc were derived in Joubert et al. (2009). In this paper, we consider the vibrating pattern of a slowly rotating solid cylindrical disc where a slight mass imperfection is introduced via a Fourier series into the equations of motion of the vibrating particles in the disc. Even though this has been investigated recently for a fluid-filled layered sphere by Shatalov et al. (2011), we feel that the dynamics of such a body are, most likely, beyond the mathematical capabilities of the target audience discussed above. In this paper, we give a step-by-step explanation of how equations of motion are derived, how a Fourier series is utilised and how the computer algebra