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On the geometry of parametrized bicubic surfaces

A. Galligo[a,*], M. Stillman[b]

[a] Laboratoire de Mathematiques, Université de Nice, 06108 Nice Cedex 02, France
[b] Department of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA

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To Carlo Traverso for his 60th birthday.

Abstract

We provide a geometric study of the problem of finding and describing the double point locus of a bicubic surface. Our motivation is to determine whether a real bicubic patch over the unit square will have self intersections. And if so, to identify useful points and curves in order to determine basic features and to help graph the surface accurately. Here, we consider special interesting cases with additional structures, which are among the surfaces commonly used in CAGD (Computer Aided Geometric Design).

Keywords: Geometry; Bicubic Bezier surface; Computer Aided Design

0. Introduction

Bicubic surface splines are widely used in applied sciences and specially in CAGD (see Hosaka (1992) and Farin (1993)). They are made from patches of parametrized bicubic surfaces defined by

$$\phi(u, v) = \left(\frac{f(u, v)}{w(u, v)}, \frac{g(u, v)}{w(u, v)}, \frac{h(u, v)}{w(u, v)}\right)$$

where f, g, h, w are all polynomials with real coefficients, each having degree 3 in u and, separately, degree 3 in v.

* Corresponding author. Tel.: +33 4 92 07 62 27; fax: +33 4 93 51 79 74.
E-mail addresses: galligo@math.unice.fr (A. Galligo), mike@math.cornell.edu (M. Stillman).

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One can see from this definition that such a surface depends on many parameters (63, in fact). It can also be described as the zero set of a homogeneous polynomial P (in the variables x, y, z, w) often of degree 18, which can have more that 500 monomials. It is not surprising that this family of surfaces has a rich geometry with possibly complicated singularities. However, as the surface is the image of a map ϕ, the singularities arising as the images of critical points of ϕ are easier to compute than the self intersections. So we will give a precise algebraic study of the double loci of the considered maps. This is a necessary step towards the full description of their topology in the real setting.

Although patches of these surfaces have been extensively used and studied, their singular loci (and particularly their double points loci) are still imperfectly known. Despite promising recent works Dokken (2001), Galligo and Pavone (in press), Galligo and Pavone (2005), Lasser (1988), Thomassen (2005), Pavone (2004) and Andersson et al. (1998), they have the reputation to be hard to compute precisely. Because, when setting the equations, one has to exclude the trivial solution where the two pairs of parameters are equal, they are harder to compute than the usual intersection of patches, for which several methods have been developed (Emiris and D'Andrea, 2002; Sederberg and Meyers, 1988; Patrikalakis, 1993; Krishnan and Manocha, 1997; Hohmeyer, 1991). Scientists and engineers in most applications choose to only use patches free of self intersections. These are hard to detect on a parametric representation. Several theoretical studies and also algorithms have aimed to delimit zones in the u, v plane where they could not appear.

However, in some applications in CAGD such as Drafting, Filling, Offsetting, singularities appear naturally and require accurate representations together with efficient computations and friendly use. Good descriptions of self-intersection loci are needed in order to construct a trimmed parametrization of the considered surfaces. Also, other families of algebraic surfaces have been studied geometrically for their use in CAGD (see e.g. Anderson and Sederberg (1985), Coffman et al. (1996), Elkadi et al. (2004) and Piene (2005)).

0.1. Our project

With this paper, we would like to contribute to the study of the geometry of bicubic spline surfaces. Our aims are mathematical descriptions as well as effective computations and eventually approximate computations (which are harder because of the possible lack of precision in the inputs).

As the singularity loci of a general such surface can be very complicated, here we choose to study special cases. These will form special families of surfaces that we will name type I, II, III and IV; they will recover surfaces commonly used in CAGD. These surfaces have additional structure which will ease the description of their geometry.

In applications, scientists and engineers only consider real patches. However, in our study, in order to apply techniques and results from algebraic geometry and computer algebra, we will also consider complex projective surfaces. This will be useful to derive information for the real affine case.

We can summarize our project by the following targets and questions.

(a) Find an algorithm to compute the double curve of a bicubic surface, and eventually also the multiple points of higher order. How bad can the singularities be? Can we detect whether all of the singularities will be far away from the real unit square?
(b) Given certain classes of bicubic surfaces, determine their geometry and algebra explicitly.