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Computing the characteristic numbers of the variety of nodal plane cubics in $\mathbb{P}^3$

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Abstract

In this note we obtain, phrased in present day geometric and computational frameworks, the characteristic numbers of the family U_{nod} of non-degenerate nodal plane cubics in $\mathbb{P}^3$, first obtained by Schubert in his *Kalkül der abzählenden Geometrie*. The main geometric contribution is a detailed study of a variety X_{nod}, which is a compactification of the family U_{nod}, including the boundary components (degenerations) and a generalization to $\mathbb{P}^3$ of a formula of Zeuthen for nodal cubics in $\mathbb{P}^2$. The computations have been carried out with the WIRIS boost WIT.

Keywords: Plane nodal curves; Characteristic numbers; Effective computational methods

0. Introduction

Given an irreducible n-dimensional family of plane curves in $\mathbb{P}^3$, we are interested in the number of curves in the family that satisfy n conditions and, in particular, in its *characteristic numbers*, namely, the number of curves that go through i given points, intersect k given lines and are tangent to $n - 2i - k$ given planes. Concerning the family of nodal cubics in $\mathbb{P}^2$, the characteristic numbers (and many other intersection numbers) were calculated by Maillard (1871), Zeuthen (1872) and Schubert (1879), and were verified, in different ways, by Sacchiero (1984), Kleiman and Speiser (1988), Aluffi (1991) and Miret and Xambó (1991).

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In this paper we study the characteristic numbers of the variety of nodal plane cubics in $\mathbb{P}^3$ given by Schubert. We first construct a compactification X_{nod} of the variety U_{nod} of non-degenerate nodal plane cubics of $\mathbb{P}^3$ by means of the projectivization of a suitable vector bundle. From this we get that the Picard group $\mathsf{Pic}(X_{\text{nod}})$ is a rank 3 free group generated by the classes μ, b and ν of the closures in X_{nod} of the hypersurfaces of U_{nod} determined, respectively, by the conditions:

– μ, that the plane determined by the nodal cubic go through a point;
– b, that the node be on a plane and
– ν, that the nodal cubic intersect a line.

We show that the boundary $X_{\text{nod}} - U_{\text{nod}}$ consists of two irreducible components of codimension 1 and we prove a formula which expresses the condition

– ρ, that the nodal cubic be tangent to a plane,

in terms of the two degenerations and the condition μ. This formula is a generalization to $\mathbb{P}^3$ of a degeneration relation given by Zeuthen (1872) for nodal cubics in the projective plane. We compute, on the basis of the intersection theory of X_{nod} and using WIT (see Xambó (2002–2006)), the intersection numbers of the form $\mu^i \nu^k \rho^{11-i-k}$ given by Schubert (1879). In particular, we get the number ν^{11} of plane nodal cubics that intersect 11 lines which was used (and verified) by Kleiman et al. (1987). Finally, the computation of the characteristic numbers $P^i \nu^k \rho^{11-2i-k}$ of the family of nodal plane cubics in $\mathbb{P}^3$ follows from the incidence formula $P = \nu\mu - 3\mu^2$, where P is the condition that the nodal cubic goes through a given point.

1. The variety X_{nod} of nodal plane cubics

In the sequel, $\mathbb{P}^3$ will denote the projective space associated to a 4-dimensional vector space over an algebraically closed ground field $\mathbf{k}$ of characteristic 0, and the term *variety* will be used to mean a quasi-projective $\mathbf{k}$-variety.

Let $\mathbb{U}$ denote the rank 3 tautological bundle over the Grassmann variety Γ of planes of $\mathbb{P}^3$. Therefore, the projective bundle $\mathbb{P}(\mathbb{U})$ is a non singular variety defined by $\mathbb{P}(\mathbb{U}) = \{(\pi, x) \in \Gamma \times \mathbb{P}^3 \mid x \in \pi\}$. Let $\mathbb{L}$ be the tautological line subbundle of the rank 3 bundle $\mathbb{U}|_{\mathbb{P}(\mathbb{U})}$ over $\mathbb{P}(\mathbb{U})$ and let $\mathbb{Q}$ be the tautological quotient bundle. We will denote by a the hyperplane class of $\mathbb{P}(\mathbb{U})$ and by μ the pullback to $\mathbb{P}(\mathbb{U})$ of $c_1(\mathcal{O}_\Gamma(1))$ under the natural projection $\mathbb{P}(\mathbb{U}) \to \Gamma$.

We define $\mathbb{E}_{\text{nod}}$ as the subbundle of $S^3\mathbb{U}^*|_{\mathbb{P}(\mathbb{U})}$ whose fiber over $(\pi, x) \in \mathbb{P}(\mathbb{U})$ is the linear subspace of forms $\varphi \in S^3\mathbb{U}^*$ defined over π that have multiplicity at least 2 at x. In fact, given a point $(\pi, x) \in \mathbb{P}(\mathbb{U})$ and taking projective coordinates x_0, x_1, x_2, x_3 so that $\pi = \{x_3 = 0\}$ and $x = [1, 0, 0, 0]$, we can express the elements φ of the fiber of $\mathbb{E}_{\text{nod}}$ over (π, x) as follows:

$$\varphi = b_1 x_0 x_1^2 + b_2 x_0 x_1 x_2 + b_3 x_0 x_2^2 + a_1 x_1^3 + a_2 x_1^2 x_2 + a_3 x_1 x_2^2 + a_4 x_2^3, \tag{1}$$

where b_i and a_i are in $\mathbf{k}$. Thus, $\mathbb{E}_{\text{nod}}$ is a rank 7 subbundle of $S^3\mathbb{U}^*|_{\mathbb{P}(\mathbb{U})}$.

In the next proposition we give a free resolution of the vector bundle $\mathbb{E}_{\text{nod}}$ over $\mathbb{P}(\mathbb{U})$. To do this, we consider the natural inclusion map $i : \mathbb{Q}^* \to \mathbb{U}^*$, the product map $\kappa : \mathbb{Q}^* \otimes S^2\mathbb{Q}^* \to S^3\mathbb{Q}^*$, and the maps

$$h : \mathbb{U}^* \otimes S^2\mathbb{Q}^* \to S^3\mathbb{U}^*|_{\mathbb{P}(\mathbb{U})} \quad \text{and} \quad j : S^3\mathbb{Q}^* \to S^3\mathbb{U}^*|_{\mathbb{P}(\mathbb{U})}$$

whose images are clearly contained in $\mathbb{E}_{\text{nod}}$.