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Achromatic axes and their linear optics

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ABSTRACT

If a polychromatic ray segment enters an optical system, is dispersed into many slightly different paths through the system, and finally emerges at a single point, then the incident segment defines what Le Grand and Ivanoff called an achromatic axis of the system. Although their ideas of some 65 years ago have inspired important work on the optics of the eye there has been no analysis of such axes for their own sake. The purpose of this paper is to supply such an analysis. Strictly speaking optical systems, with some exceptions, do not have achromatic axes of the Le Grand–Ivanoff type. However, achromatic axes based on a weaker definition do exist and may for practical purposes, perhaps, be equivalent to strict Le Grand–Ivanoff axes. They are based on a dichromatic incident ray segment instead. The linear optics of such achromatic axes is developed for systems, like the visual optical system of the eye, that may be heterocentric and astigmatic. Equations are obtained that determine existence and uniqueness of the axes and their locations. They apply to optical systems like the eye and the eye in combination with an optical instrument in front of it. Numerical examples involving a four-refracting surface eye are treated in Appendix A. It has a unique achromatic axis for each retinal point including the center of the fovea in particular. The expectation is that the same is true of most eyes. It is natural to regard the Le Grand–Ivanoff achromatic axis as one of a class of six types of achromatic axes. A table lists formulae for locating them.

1. Introduction

In a footnote Ivanoff (1946a) mentioned Le Grand's view that 'one should not talk of optical axis, which is nothing but a fiction, the eye not being a centered system, but of what one could call *achromatic axis*, an axis such that the rays of various wavelengths penetrating the eye along this axis all encounter the retina at the same point, after having followed trajectories that were possibly slightly different. Such an *achromatic axis* is a physical reality'. (Translation by W. F. H. from the French.) The concept arose in Ivanoff's work on chromatic and spherical aberration (Ivanoff, 1946a, 1946b, 1947, 1950, 1953a, 1953b, 1956a, 1956b) and inspired much valuable work (e.g. Bradley & Thibos, 1995; Bradley, Zhang, & Thibos, 1991; Howarth, 1984; Howarth & Bradley, 1986; Jóźwicki, 1965; Kruger, López-Gil, & Stark, 2001; Marcos et al., 1999; Simonet & Campbell, 1990a, 1990b; Thibos & Bradley, 1999; Thibos et al., 1990, 1992; Woods, Bradley, & Atchison, 1996) on the optics of the eye. In particular it inspired the Thibos–Bradley concept (Bradley & Thibos, 1995; Thibos & Bradley, 1999; Thibos et al., 1990) of the achromatic axis. (Theirs is an aperture-dependent concept whereas the Le Grand-Ivanoff definition is aperture independent.) Although it is the Thibos–Bradley achromatic axis which seems to be more familiar with researchers today both it and the Le Grand–Ivanoff concept are encountered in the literature (e.g. Atchison & Smith, 2000, p. 32). Under the entry 'achromatic axis' two modern dictionaries (Millodot, 2009, p. 38; Millodot & Laby, 2002, p. 29) describe the Le Grand–Ivanoff concept while a third (Hofstetter et al., 2000, p. 43) uses the Thibos–Bradley concept. Yet a third concept also goes by the name achromatic axis: it is an axis in color space (e.g. Beer & MacLeod, 2000; Cohen, 2001, pp. 66 et seq.). Neither the Thibos–Bradley concept nor axes in color space are considered in this paper.

Despite its importance all references in the literature to the Le Grand–Ivanoff achromatic axis have been brief. Furthermore, interpretations of the concept vary; e.g. Koonen, Scolnik, and Tousey (1956) write 'For reference axis Ivanoff chose the "achromatic axis," which is the ray to the center of the fovea entering the eye at a point such that there is no dispersion'. This attributes to Ivanoff three things not in his definition quoted above: that the achromatic axis is a particular ray (it is the infinite straight line containing only the *incident segment* of the ray); that the common point on the retina is at the center of the fovea (it may be anywhere on the retina); and that there is no dispersion (there usually is dispersion of the type called *inclinational dispersion* in this paper). Part of the problem may be that there has never been a study of the concept for its own sake. Accordingly the purpose of this paper is to attempt to satisfy that want. More particularly the objective

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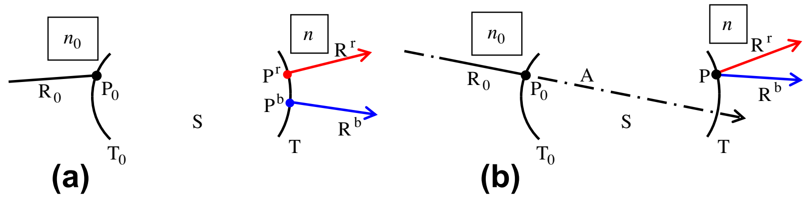

Fig. 1. A ray with dichromatic incident segment R_0 traverses arbitrary system S and emerges as two separate segments R^r and R^b in (a). In (b) the emergent segments emerge at the same point P. The directed straight line A in (b), defined by the incident segment R_0, is a Le Grand–Ivanoff achromatic axis of S.

here is to develop the linear optics of the Le Grand–Ivanoff achromatic axis. For the most part, however, we shall be considering dichromatic light instead of the polychromatic light implied in Ivanoff's definition.

We begin with the dichromatic segment of a ray incident onto an optical system. Within the system the two components (called 'red' and 'blue' for convenience), because of chromatic dispersion, may follow slightly different paths and emerge from the system at different points. Linear optics provides equations that govern this behavior. We then impose the condition required by Ivanoff's (1946a) definition that the two rays emerge at the same point. The result is equations for the position and direction of the incident dichromatic segment. The equations enable us to examine existence, uniqueness, and location of achromatic axes in the system.

We shall consider optical systems in general although we often have the visual optical system of the eye in mind. The systems may be heterocentric and astigmatic. The analysis will be specialized to particular systems including the eye and the reduced eye. Numerical examples in Appendix A calculate the locations of achromatic axes, including the centro-foveal achromatic axis, in a model eye with four separated, tilted, and astigmatic surfaces.

Related to the Le Grand–Ivanoff achromatic axis are other axes that are also achromatic but in different senses. The complete set represents a natural generalization of the Le Grand–Ivanoff axis and each can be analyzed by methods similar to those used below for the Le Grand–Ivanoff axis. For sake of completeness they are described briefly towards the end of the paper and equations for locating them are presented in a table.

2. Achromatic axes

Fig. 1 represents an arbitrary optical system S that lies between entrance surface T_0 and exit surface T. The index of refraction before T_0 is n_0 and the index after T is n. Incident onto S is the dichromatic segment R_0 of a ray. Because of dispersion taking place at interfaces within S the ray emerges as two separated segments R^r and R^b in Fig. 1a. We shall refer to the emergent segments as red and blue[1] respectively, as suggested by the superscripts, but they may be of any distinct frequencies. Fig 1a represents the usual situation. Fig. 1b represents a situation that is special in that R^r and R^b emerge at the *same* point P in exit surface T. The incident segment R_0 then defines an *achromatic axis* A of system S for the two particular frequencies. Within system S we expect the red and blue rays to follow slightly different paths but our concern will be only with the positions and directions of rays at incidence onto and emergence from S. Achromatic axis A is a straight line, infinite in both senses along it. It contains the incident segments of the rays but does not necessarily bear any simple relationship to the emergent segments. In particular, as suggested by Fig. 1b, it need not intersect the point of emergence.

[1] For interpretation of color in Figs. 1 and 4, the reader is referred to the web version of this article.

'Achromatic axis', as defined here, requires only that rays of just the two frequencies emerge at point P. Our definition is effectively a relaxed version of Ivanoff's (1946a) which implies the full visible spectrum. If incident segment R_0 were polychromatic one might suppose that rays of intermediate frequencies would emerge as segments between the red and blue segments in Fig. 1 and, hence, that all emerge at the same point P in Fig. 1b. In that case, our definition would be equivalent to Ivanoff's. However, as we shall see, this turns out to be true only for some special systems, including reduced eyes. For more complicated systems, including most eyes, rays of intermediate frequencies emerge close to but not strictly at point P. However the distance from P is probably sufficiently small to be negligible for most practical purposes.

An achromatic axis of system S is special for S in that a dichromatic segment incident along it emerges with no positional dispersion. Nothing is implied so far, however, about whether an achromatic axis actually exists for S or is unique. These are issues we shall be examining below.

We turn now to the problem of locating an achromatic axis in an optical system. We shall aim to locate the incident point P_0 and the direction of the incident segment there. In order to allow for astigmatism and heterocentricity, we shall work in three dimensions rather than two implied in Fig. 1.

3. Linear optics

Fig. 2 represents an optical system S. Z is a longitudinal axis. The entrance and exit surfaces are now entrance and exit planes T_0 and T respectively. For more detail on the symbolism and the linear optics which we now invoke the reader is referred to earlier papers (e.g. Harris, 2010a, 2011a, 2011b). R_0 and R are the incident and emergent segments of a ray of a particular frequency ν. R_0 is incident onto S at transverse position $\mathbf{y}_0$ in T_0 and with inclination $\mathbf{a}_0$. $\mathbf{y}_0$ and $\mathbf{a}_0$ are the incident position and inclination of the ray respectively. Similarly the emergent segment R has emergent position $\mathbf{y}$ and emergent inclination $\mathbf{a}$. Each of $\mathbf{y}_0$, $\mathbf{y}$, $\mathbf{a}_0$, and $\mathbf{a}$ is a 2×1 matrix representing Cartesian coordinates with respect to axis Z. We write the magnitudes of these vectors as y_0, y, a_0, and a respectively.

Refractive indices n_0 immediately upstream and n immediately downstream of system S define what we refer to as the *context* of S. What happens to a ray traversing S depends not only on S itself but also its context. Reference to a system from here on implies the system within its particular context.

Our objective below will be to attempt, using linear optics, to locate achromatic axes, if any, in a given optical system S. That will involve obtaining the incident position $\mathbf{y}_0$ or inclination $\mathbf{a}_0$, or both, of a ray that satisfies the definition. It turns out that, in each case, the problem boils down to that of solving an equation of the form

$$\mathbf{Gx} = \mathbf{d} \qquad (1)$$

for an unknown $m \times 1$ matrix $\mathbf{x}$. $\mathbf{G}$ is a known $m \times m$ matrix and $\mathbf{d}$ a known $m \times 1$ matrix. m may be 2 or 4. $\mathbf{G}$ and $\mathbf{d}$ are properties of S within its context, and $\mathbf{x}$ defines the incident position or direction