Knowledge-Based Systems 99 (2016) 92–102

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

The attribute reductions of three-way concept lattices

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ARTICLE INFO

Article history:
Received 25 January 2015
Revised 8 January 2016
Accepted 30 January 2016
Available online 18 February 2016

Keywords:
Three-way concept
Attribute reduction
Irreducible element
Object-induced three-way object concept
Attribute-induced three-way attribute concept
Discernibility attribute set

ABSTRACT

Three-way concept analysis is a newly proposed area of formal concept analysis from which one can obtain both the inclusion decision and the exclusion decision. In general, given a context, some attributes may not be essential in three-way concept analysis, such as forming three-way concept lattice. So in this paper, we study the attribute reductions of three-way concept lattices in order to make the data easily be understood. Firstly, based on different criteria generated from object-induced three-way concept (OE-concept), four kinds of attribute reductions are proposed. The four reductions together embody different characteristics of a formal context and can be used in different occasions. Secondly, we discuss their relationships, including their advantages and disadvantages and the relationships among consistent sets and among the cores. Thirdly, based on attribute-induced three-way concept (AE-concept), we also give four attribute-induced three-way attribute reductions and discuss their relationships. Finally, the approaches to computing these attribute reductions are presented and the obtained results are demonstrated and verified by an empirical case. In this paper, we systematically investigate the attribute reductions of three-way concept lattices which enriches the study of formal concept analysis.

1. Introduction

The three-way decision (3WD), an extension of the two-way decision model with an added third option, is a common human practice in problem solving and is widely used in our daily life's decision making process [1]. Because of its extensive usages, Yao [2] outlined the theory of three-way decision. The basic idea of 3WD is to divide the universe of discourse into three disjoint subsets by given criteria. These three disjoint parts are called positive, negative and boundary regions from which one can construct different rules to make three-way decision. One can construct rules of acceptance from positive region, rules of rejection from negative region and rules of non-commitment from boundary region.

In the past few years, more and more attentions have been paid to this newly proposed theory and the 3WD has been studied from the different views. Actually, 3WD plays a key role in knowledge-based systems and has been widely used in many fields and disciplines, including computer science, information science, management science, engineering, social science, medical decision-making, etc.[1,3–13].

Formal concept analysis (FCA), as an efficient tool for decision making and knowledge discovery, was proposed by Wille in 1982 [14], and has subsequently been extended by Wille and Ganter [15] and other scholars. As an effective method for data analysis and knowledge processing, FCA has been used in various areas, such as data mining, information retrieval, and software engineering [16–22].

However, in FCA we can only get the two-way decision rather than the three-way decision. That is, based on a formal concept, one can only determine whether an object (attribute) certainly possesses (is shared by) all elements in the intent (extent) and such decision is called inclusion method. Actually, the formal context also offers us the information about whether an object (attribute) does not possesses (is not shared by) any attribute (object), which is called exclusion method, but this cannot be reflected in the formal concept. Since this kind of exclusion method is commonly used in our daily life, Qi et al. [23] applied 3WD to FCA and proposed three-way concept analysis (3WCA). Two key components in 3WCA are three-way concepts and three-way concept lattices. Similar with the formal concept in FCA, a three-way concept is also determined by two parts named extent and intent. However, the extent or intent of a three-way concept should be an orthopair studied by Ciucci [24]. That is to say, in 3WCA, "the extent (or the intent) of a three way concept is equipped with two parts: positive one and negative one. These two parts are used to express the semantics 'jointly possessed' and 'jointly not possessed' in a formal context." [25] This newly proposed three-way concept combines inclusion method (positive region) with exclusion method (negative region). Thus based on a three-way concept, one can divide

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http://dx.doi.org/10.1016/j.knosys.2016.01.045
0950-7051/

the object (attribute) universe into three regions to make three-way decision.

Being an important issue in knowledge discovery, attribute reduction has been extensively studied in different fields of soft computing since it can decrease the dimension and make the data easily be understood [26–35]. It is also an interesting topic in FCA. Many significant results in attribute reduction have been obtained based on a formal context. For example, Zhang et al. [36] discussed the lattice reduction theory where the notion of "reducing the context" in [15] was developed. Within such framework, Zhang et al. [36] constructed judgment theorems of consistent sets, and developed approaches to attribute reduction based on discernibility matrix, which was given a further simplification by Qi [37] from the viewpoint of parent-child concepts. Based on the lattice reduction theory of formal contexts, Wei et al. [38] also studied the reduction theory of formal decision contexts under two different meanings of consistency. And Li et al. [39] have proposed the attribute reduction in order to preserve the rules in the formal decision context. On the other hand, Wang and Ma [40] proposed another attribute reduction in order to preserve the extents of meet-irreducible elements in the original formal concept lattice. Based on the irreducible elements, Li et al. [41] defined the join-irreducible attribute reduction in order to preserve the extents of join-irreducible elements in the original formal concept lattice. From the viewpoint of granular computing, granular reduction was proposed by Wu et al. [42] Additionally, Liu et al. [43] have studied the attribute reduction of object oriented formal concept lattices and property oriented formal concept lattices, and furthermore Wang et al. [44] discussed relations of attribute reductions between object and property oriented formal concept lattices.

Because of the extensive usages of 3WD and FCA, 3WCA will become a very useful tool for knowledge discovery and data analysis. In order to make the discovery and representation of implicit knowledge in three-way concept lattices easier and simpler, based on the different criteria generated from three-way concepts, we firstly propose different attribute reductions. Since the generalization and the specialization of the knowledge contained in the context are all reflected in the three-way concept lattices, if our reduction can preserve the lattice structure, then the basic knowledge reflected in the three-way concept lattices is preserved. Also, because every three-way concept is the join (meet) of join (meet)-irreducible elements, if our reduction can preserve the irreducible elements, we actually preserve the basic elements in lattice construction. Thus this kind of reduction is very important in lattice construction. From the perspective of granular computation, a three-way object or attribute concept can be regarded as a kind of granule, so a kind of attribute reduction which can preserve the extents of the three-way object (attribute) concepts is proposed. All these attribute reductions together embody different characteristics of a context. The basic ideas of these attribute reductions are similar with the classical attribute reductions mentioned in the previous paragraph. However, different from the attribute reductions in FCA, all the reductions in this paper depend on the three-way concept and the three-way concept can reflect more information than the classical formal concept, so the results and meaning of these attribute reductions in 3WCA are different with the classical ones in FCA. Then, we analyze the relationships among these three-way reductions. Finally, we give some approaches to these attribute reductions. Since the study we make is based on the three-way concept analysis, our work can be regarded as a three-way extension of the attribute reductions mentioned in previous paragraph.

This paper mainly discusses the different attribute reductions of three-way concept lattices and the relationships among them. The content is organized as follows. In Section 2, the basic knowledge of classical and three-way concept analysis is reviewed. Section 3 gives the definitions of four different OE-attribute reductions and discusses their relationships, including the relationships among the corresponding consistent sets and cores. Analogously, in Section 4, we define four different AE-attribute reductions and discuss the relationships among them. Then, the approaches to computing these attribute reductions are presented in Section 5. Finally, an empirical case is shown in Section 6 and this paper is concluded with a summary in Section 7.

2. Preliminaries

In order to make this paper self-contained, we firstly review some basic notions in formal concept analysis and three-way concept analysis.

2.1. Formal concept analysis

Definition 1 [15]**.** A formal context (G, M, I) consists of two sets G and M, and a relation I between G and M. The elements of G are called the objects and the elements of M are called the attributes of the context. In order to express that an object x is in a relation I with an attribute m, we write xIm or $(x, m) \in I$ and read it as "the object x has the attribute m".

A pair of operators are defined on $X \subseteq G$ and $A \subseteq M$ respectively in (G, M, I) [15]:

$$X^* = \{m | m \in M, \forall x \in X, xIm\}$$
$$A^* = \{x | x \in G, \forall m \in A, xIm\}$$

The operator $*$ is same with the modal-style operator $R^{\triangle}$ in [45], which expresses the meaning of jointly possessing.

The properties of this pair of operators are shown in [15], so we do not describe here.

Definition 2 [15]**.** A formal concept of the context (G, M, I) is a pair (X, A) with $X^* = A$ and $X = A^*(X \subseteq G, A \subseteq M)$. We call X the extent and A the intent of the formal concept (X, A).

The formal concepts of a formal context (G, M, I) are ordered by

$$(X_1, A_1) \leq (X_2, A_2) \Leftrightarrow X_1 \subseteq X_2 \ (\Leftrightarrow A_1 \supseteq A_2).$$

All formal concepts of (G, M, I) can form a complete lattice called the formal concept lattice of (G, M, I) and denoted by $L(G, M, I)$. The infimum and supremum are given by

$$(X_1, A_1) \wedge (X_2, A_2) = (X_1 \cap X_2, (A_1 \cup A_2)^{**})$$

$$(X_1, A_1) \vee (X_2, A_2) = ((X_1 \cup X_2)^{**}, A_1 \cap A_2)$$

In order to distinguish from the following three-way concept analysis, the formal concept mentioned above will be called classical concept and other notions in the FCA also be named in the same way in the following parts of this paper.

Now, we give the definition of reduct as follows.

Definition 3 [36]**.** Let (G, M, I) be a formal context. If there exists an attribute set $D \subseteq M$ such that $L(G, D, I_D) \cong L(G, M, I)$, then D is called a consistent set of (G, M, I). And further, if $\forall d \in D, L(G, D - \{d\}, I_{D-\{d\}}) \ncong L(G, M, I)$, the set D is called a reduct of (G, M, I). The intersection of all the reducts is called the core of (G, M, I). Here $I_D = I \cap (G \times D)$.

Definition 4 [46]**.** Let L be a lattice. An element $x \in L$ is join-irreducible if

1. $x \neq 0$ (in case L has a zero)
2. $x = a \vee b$ implies $x = a$ or $x = b$ for all $a, b \in L$.

A meet-irreducible element is dually defined.