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On the axiomatization of some classes of discrete universal integrals

Erich Peter Klement [a,*], Radko Mesiar [b,c]

[a] Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, Linz, Austria
[b] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia
[c] Institute of Theory of Information and Automation, Czech Academy of Sciences, Prague, Czech Republic

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ABSTRACT

Following the ideas of the axiomatic characterization of the Choquet integral due to [D. Schmeidler, Integral representation without additivity, Proc. Amer. Math. Soc. 97 (1986) 255–261] and of the Sugeno integral given in [J.-L. Marichal, An axiomatic approach of the discrete Sugeno integral as a tool to aggregate interacting criteria in a qualitative framework, IEEE Trans. Fuzzy Syst. 9 (2001) 164–172], we provide a general axiomatization of some classes of discrete universal integrals, including the case of discrete copula-based universal integrals (as usual, the product copula corresponds just to the Choquet integral, and the minimum to the Sugeno integral).

1. Introduction

In this contribution, we restrict ourselves to a fixed finite space $X = \{1,\dots,n\}$, and we will deal with functions from X to $[0,1]$ which we identify with n-dimensional vectors $\mathbf{x} = (x_1,\dots,x_n)$. From the application point of view, we can look at $\mathbf{x}$ as a score vector of some alternative characterized by n criteria. To be able to decide which of the alternatives described by the score vectors $\mathbf{x}$ and $\mathbf{y}$, respectively, is to be preferred, a typical approach is to evaluate both $\mathbf{x}$ and $\mathbf{y}$ by means of some utility function U.

The utility function U is often constructed from a boolean utility function B acting on $\mathbf{x} \in \{0,1\}^n$. However, each such boolean utility function B can be seen as a capacity $m : 2^X \to [0,1]$, $m(E) = B(\mathbf{1}_E)$. Typical extension approaches are related to integration, i.e., $U(\mathbf{x}) = \mathbf{I}(m,\mathbf{x})$, where $\mathbf{I}(m,\cdot)$ is some integral on X with respect to the capacity m.

Another approach is based on some axiomatization (and boolean utility function B). It is well-known that the additivity of the utility function $U : [0,1]^n \to [0,1]$ is related to the application of Lebesgue integral, $U(\mathbf{x}) = \int \mathbf{x}\, dm$, and then also the capacity m should be additive. Putting $m(\{i\}) = w_i$, we obtain the well-known weighted arithmetic mean, $U(\mathbf{x}) = \sum_{i=1}^{n} w_i \cdot x_i$.

Our contribution recalls some classes of universal integrals (including, among others, the Choquet, the Sugeno and the Lebesgue integral) and provides corresponding axiomatizations.

Because of the link to utility functions, we restrict ourselves to (normed) capacities and to input values from $[0,1]$, although many integrals mentioned here (including the Choquet and Sugeno integral) can be considered in a more general (unbounded) framework [14].

However, we do not consider any further restriction concerning the underlying capacity, such as additivity or pseudo-additivity, and thus we will not deal with integrals based on such special capacities (compare, e.g., [20,22,27]).

The paper is organized as follows. In the following section, the Choquet and the Sugeno integral as well as their axiomatizations are summarized. In Section 3, we recall (discrete) copula-based integrals and some other classes of discrete universal integrals, including some examples. In Section 4, the axiomatization of these discrete universal integrals is given. As a special case, symmetric discrete copula-based universal integrals (generalizing OWA operators) are discussed.

2. Choquet and Sugeno integrals, and their axiomatization

Though all integrals discussed in this paper can be defined on an arbitrary measurable space, in this paper we consider (as already mentioned) the finite space $X = \{1,\dots,n\}$ only, equipped with the σ-algebra $2^X = \{E | E \subseteq X\}$.

Definition 2.1. A *capacity* on X is a set function $m : 2^X \to [0,1]$ which is non-decreasing, i.e., we have $m(E) \leqslant m(F)$ whenever $E \subseteq F \subseteq X$, and satisfies the boundary conditions $m(\emptyset) = 0$ and $m(X) = 1$.

* Corresponding author.
E-mail addresses: ep.klement@jku.at (E.P. Klement), mesiar@math.sk (R. Mesiar).

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Then the *Choquet integral* [3] of $\mathbf{x}$ with respect to a capacity $m : 2^X \to [0,1]$ is defined by

$$\mathbf{Ch}(m,\mathbf{x}) = \int_0^1 m(\{i|x_i \geqslant t\})\, dt$$
$$= \sum_{i=1}^{n} x_{\pi_i} \cdot (m(\{\pi_i,\ldots,\pi_n\}) - m(\{\pi_{i+1},\ldots,\pi_n\})), \tag{1}$$

for some permutation $(\pi_1,\pi_2,\ldots,\pi_n)$ of $\{1,\ldots,n\}$ satisfying $x_{\pi_1} \leqslant x_{\pi_2} \leqslant \cdots \leqslant x_{\pi_n}$, where the set $\{\pi_{n+1},\pi_n\}$ occurring in the last summand is defined to be the empty set $\emptyset$.

Obviously, we have $m(E) = \mathbf{Ch}(m,\mathbf{1}_E)$ for each $E \subseteq X$. Observe that if m is additive (i.e., m is a discrete probability measure) then the Choquet integral coincides with the Lebesgue integral, i.e., $\mathbf{Ch}(m,\mathbf{x}) = \int \mathbf{x}\, dm$.

Similarly, the *Sugeno integral* [26] of $\mathbf{x}$ with respect to a capacity $m : 2^X \to [0,1]$ is given by

$$\mathbf{Su}(m,\mathbf{x}) = \bigvee_{t=0}^{1}(t \wedge m(\{i|x_i \geqslant t\})) = \bigvee_{i=1}^{n}(x_{\pi_i} \wedge m(\{\pi_i,\ldots,\pi_n\})). \tag{2}$$

Note that we use the symbols $\wedge$ and $\vee$ in the sense $x \wedge y = \min(x,y)$ and $x \vee y = \max(x,y)$. Clearly, also for the Sugeno integral we have $m(E) = \mathbf{Su}(\mathbf{1}_E)$ for all $E \subseteq X$.

In what follows, the comonotonicity of score vectors plays a crucial role.

Definition 2.2 [23]. Let $\mathbf{x}$, $\mathbf{y} \in [0,1]^n$. Then $\mathbf{x}$ and $\mathbf{y}$ are said to be *comonotone* if, for all i, $j \in \{1,2,\ldots,n\}$, we have $(x_i - x_j) \cdot (y_i - y_j) \geqslant 0$.

In other words, for comonotone $\mathbf{x}$, $\mathbf{y} \in [0,1]^n$ it is impossible to have $x_i > x_j$ and $y_i < y_j$. In [23] an axiomatic characterization of the Choquet integral as a comonotone aggregation function [4,8] was given.

Definition 2.3

(i) An (n-dimensional) *aggregation function* is a function $A : [0,1]^n \to [0,1]$ which is non-decreasing in each component and satisfies the boundary conditions $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$.
(ii) An aggregation function $A : [0,1]^n \to [0,1]$ is said to be *comonotone additive* if, for all $\mathbf{x}$, $\mathbf{y} \in [0,1]^n$ which are comonotone and satisfy $\mathbf{x} + \mathbf{y} \in [0,1]^n$, we have
$$U(\mathbf{x}+\mathbf{y}) = U(\mathbf{x}) + U(\mathbf{y}).$$

Observe that the comonotone additivity of an aggregation function U implies its positive homogeneity, i.e., $U(c \cdot \mathbf{x}) = c \cdot U(\mathbf{x})$ for all $c \geqslant 0$ and $\mathbf{x} \in [0,1]^n$ with $c \cdot \mathbf{x} \in [0,1]^n$.

Proposition 2.4 [23]. *Let $U : [0,1]^n \to [0,1]$ be an n-ary aggregation function. Then the following are equivalent:*

(i) *There is a capacity $m : 2^X \to [0,1]$ such that $U(\cdot) = \mathbf{Ch}(m,\cdot)$.*
(ii) *U is comonotone additive.*

In the case of Sugeno integral, its axiomatization was given in [16].

Proposition 2.5 [16]. *Let $U : [0,1]^n \to [0,1]$ be an n-ary aggregation function. Then the following are equivalent:*

(i) *There is a capacity $m : 2^X \to [0,1]$ such that $U(\cdot) = \mathbf{Su}(m,\cdot)$.*
(ii) *U is $\wedge$-homogeneous and comonotone maxitive, i.e., for each $c \in [0,1]$, the constant score vector $\mathbf{c} = (c,\ldots,c)$ and all comonotone $\mathbf{x}$, $\mathbf{y} \in [0,1]^n$ we have*
$$U(\mathbf{c} \wedge \mathbf{x}) = c \wedge U(\mathbf{x}),$$
$$U(\mathbf{x} \vee \mathbf{y}) = U(\mathbf{x}) \vee U(\mathbf{y}).$$

Observe that the comonotone maxitivity of an aggregation function U does not imply its $\wedge$-homogeneity. Note that there are some alternative axiomatic approaches to the Sugeno integral (compare [1,16]).

3. Some classes of discrete universal integrals

We briefly recall some classes of discrete universal integrals which will be characterized in an axiomatic way in Section 4. For functions with values in the nonnegative real numbers, the concept of a *universal integral* which can be defined on arbitrary (not necessarily finite) measurable spaces and for arbitrary capacities, was introduced axiomatically and investigated in [14]. It is based on a special type of binary aggregation function, the so-called semicopula [5].

Definition 3.1. A *semicopula* is two-dimensional aggregation function $\odot : [0,1]^2 \to [0,1]$ with neutral element 1.

Given a semicopula $\odot[0,1]^2 \to [0,1]$ and a capacity $m: \to [0,1]$ we will require that each discrete universal integral acts on $[0,1]^n$ as a special aggregation function.

Definition 3.2. Let $\odot : [0,1]^2 \to [0,1]$ be a semicopula and let $m : 2^X \to [0,1]$ be a capacity on X. A *discrete universal integral* (*based on* $\odot$) is an aggregation function $\mathbf{I}_{\odot,m} : [0,1]^n \to [0,1]$ such that

(i) for all $c \in [0,1]$ and all $E \subseteq X$ we have $\mathbf{I}_{\odot,m}(c \cdot \mathbf{1}_E) = c \odot m(E)$;
(ii) for all $\mathbf{x}$, $\mathbf{y} \in [0,1]^n$ with $m(\{i \in X|x_i \geqslant t\}) = m(\{j \in X|y_j \geqslant t\})$ for all $t \in [0,1]$ we have $\mathbf{I}_{\odot,m}(\mathbf{x}) = \mathbf{I}_{\odot,m}(\mathbf{y})$.

Note that each discrete universal integral as given in Definition 3.2 is an idempotent aggregation function because of

$$\mathbf{I}_{\odot,m}(c,c,\ldots,c) = \mathbf{I}_{\odot,m}(c \cdot \mathbf{1}_X) = c \odot m(X) = c \odot 1 = c.$$

Observe that if a capacity m assumes values in $\{0,1\}$ only then all discrete universal integrals are independent of the semicopula $\odot$, and they correspond to lattice polynomials (compare [15]). Moreover, the class of discrete universal integrals is convex, i.e., for each monotone measure m, for all discrete universal integrals $\mathbf{I}^{(1)}_{\odot_1,m}$ and $\mathbf{I}^{(2)}_{\odot_2,m}$ based on the semicopulas $\odot_1$ and $\odot_2$, respectively, and for each $\lambda \in [0,1]$, also

$$\mathbf{I}_{\odot,m} = \lambda \cdot \mathbf{I}^{(1)}_{\odot_1,m} + (1-\lambda) \cdot \mathbf{I}^{(2)}_{\odot_2,m}$$

is a discrete universal integral based on the semicopula $\odot = \lambda \cdot \odot_1 + (1-\lambda) \cdot \odot_2$.

3.1. Discrete copula-based universal integrals

Universal integrals (acting on the interval $[0,\infty]$) were introduced and discussed in [14]. A special kind of universal integrals on the scale $[0,1]$ is based on copulas [21,25], compare also [13].

Definition 3.3. A (*binary*) *copula* $C : [0,1]^2 \to [0,1]$ is a semicopula which is supermodular, i.e., for all $\mathbf{x}$, $\mathbf{y} \in [0,1]^2$

$$C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geqslant C(\mathbf{x}) + C(\mathbf{y}). \tag{3}$$

We are not going into details about universal integrals and copulas here, we only recall the following important result (see Remark 5.3, 2 in [14]):

Proposition 3.4 [14]. *Let $C : [0,1]^2 \to [0,1]$ be a copula and $m : 2^X \to [0,1]$ a capacity, and define $\mathbf{K}_C(m,\cdot) : [0,1]^n \to [0,1]$ by*