Knowledge-Based Systems 22 (2009) 449–454

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

Entropy of interval-valued fuzzy sets based on distance and its relationship with similarity measure

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ARTICLE INFO

Article history:
Received 27 February 2008
Accepted 4 June 2009
Available online 12 June 2009

Keywords:
Interval-valued fuzzy sets
Entropy
Similarity measure
Intuitionistic fuzzy sets
Measures of information

ABSTRACT

This article proposes a new axiomatic definition of entropy of interval-valued fuzzy sets (IVFSs) and discusses its relation with similarity measure. First, we propose an axiomatic definition of entropy for IVFS based on distance which is consistent with the axiomatic definition of entropy of a fuzzy set introduced by De Luca, Termini and Liu. Next, some formulae are derived to calculate this kind of entropy. Furthermore we investigate the relationship between entropy and similarity measure of IVFSs and prove that similarity measure can be transformed by entropy. Finally, a numerical example is given to show that the proposed entropy measures are more reasonable and reliable for representing the degree of fuzziness of an IVFS.

1. Introduction

Interval-valued fuzzy sets [24,26,34] and intuitionistic fuzzy sets (IFSs) [1] are two intuitively straightforward extensions of Zadeh's fuzzy set [33]. These two fuzzy sets were independently introduced to alleviate some drawbacks of fuzzy set. IVFS theory emerges from the observation that in a lot of cases, no objective procedure is available to select the crisp membership degrees of elements in a fuzzy set. It was suggested to specify an interval-valued degree of membership $[\mu_1, \mu_2]$ to each element of the universe. An IFSs allocates to each element of the universe both a degree of membership μ and one of non-membership v such that $\mu + v \leqslant 1$, thus relaxing the enforced condition $v = 1 - \mu$ from fuzzy set theory. IVFS and IFS theory have the virtue of complementing fuzzy sets, which are able to model vagueness and uncertainty precisely. They have been used in different fields of science, for example, Sambuc [24] in medical diagnosis in thyroidian pathology; Gorzalczany [12] and Bustince [6] in approximate reasoning; Turksen [27] and Cornelis et al. [7] in interval-valued and intuitionistic logic, etc.

Entropy and similarity measure of fuzzy sets are two important topics in fuzzy set theory. The entropy of fuzzy set describes the fuzziness degree of fuzzy set. Zadeh [32] first introduced the fuzzy entropy in 1965. De Luca and Termini [19] suggested the axiomatic construction of entropy of fuzzy sets and referred to Shannon's probability entropy, interpreting it as a measure of the amount of information. Kaufmann [15] pointed out that an entropy of a fuzzy set can be gotten through the distance between the fuzzy set and it's nearest non-fuzzy set, whereas Yager [31] did it using the distance from a fuzzy set to its complement. Loo [18] proposed an entropy which includes those given by De Luca and Termini [19] and Zadeh [34]. Liu [17] gave the well-known axiomatic definitions of fuzzy similarity, distance and entropy. Mi et al. [20] introduced a generalized axiomatic definition of entropy of a fuzzy set with a distance based on the axiomatic definition of Liu [17]. On the other hand, similarity measure of fuzzy sets indicates the similarity degree of fuzzy sets and has received much more attention than entropy and the corresponding literature is very extensive. Pappis and his collaborators have issued a series of papers [21–23] which took an axiomatic view of similarity measures. In [11], Fan proposed an alternative set of axioms for similarity measure. Wang et al. [28] criticized Pappis' work and presented a modified definition of similarity. Wang [29] adopted Liu's similarity axioms [17] and introduced two new similarity measures.

Some authors have investigated entropy and similarity measure of IVFSs and IFSs. Burillo and Bustince [5] introduced the concept of entropy of IFSs, which allows us to measure the degree of intuitionism of an IFS; Szmidt and Kacprzyk [25] proposed a nonprobabilistic-type entropy measure with a geometric interpretation of IFSs; Zeng and Li [35] expressed the axioms of Szmidt and Kacprzyk using the notion of IVFSs and discussed the relationship between similarity measure and entropy. Hung and Yang [14] proposed their axiom definition of entropy of IFSs and IVFSs by exploiting the concept of probability. Zhang and Fu [36] discussed the similarity measures on L-fuzzy sets. Li et al. [16] gave a

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doi:10.1016/j.knosys.2009.06.007

comparative analysis of similarity measures between IFSs and demonstrated the positive aspects of each similarity measure.

The rest of this article is organized as follows. In Section 2, we recall some basic notions of IVFSs and IFSs. In Section 3, we introduce the concepts of distance and similarity measure of IVFSs and propose a new axiomatic definition of entropy for IVFSs. Some formulae are derived to calculate this kind of entropy. In Section 4, we investigate relationship between the entropy and similarity measure of IVFSs and prove some theorems that entropy and similarity measure of IVFSs can be transformed by each other based on the axiomatic definitions. In Section 5, a numerical example is given to compare the entropy in [5,14,25,35] with our proposed entropy on IVFSs. The final section is conclusion.

2. Interval-valued fuzzy sets and intuitionistic fuzzy sets

Throughout this paper, we write $X = \{x_1, \ldots, x_n\}$ to denote the discourse set; $\mathscr{P}(X)$ and $\mathscr{F}(X)$ stands for the sets of all the crisp sets and fuzzy sets in X, respectively; $D([0,1])$ expresses the set of all the closed subintervals of the interval [0,1]. $[\frac{1}{2}]_X$ is the fuzzy set of X for which $[\frac{1}{2}]_X(x) = \frac{1}{2}$ for all $x \in X; A^c$ is the complement of fuzzy set A where $A^c(x) = 1 - A(x)$.

2.1. Definition, operations on IVFSs and IFSs

Definition 1 [24]. An interval-valued fuzzy set in X is an expression A denoted by

$$A = \{\langle x, A(x)\rangle | x \in X\},$$

where

$$A : X \to D([0,1])$$

$$x \to A(x) = [\underline{A}(x), \overline{A}(x)] \in D([0,1]).$$

For simplicity, we write $A = [\underline{A}, \overline{A}]$. $IVFSs(X)$ stands for the set of all the IVFSs in X; $[a,b]_X$ is the IVFS of X for which $[a,b]_X(x) = [a,b], \forall x \in X$.

The following expressions are defined in [24] for all $A, B \in IVFSs(X)$:

(1) $A \leqslant B$ if and only if $\underline{A}(x) \leqslant \underline{B}(x)$ and $\overline{A}(x) \leqslant \overline{B}(x)$ for all $x \in X$;
(2) $A = B$ if and only if $A \leqslant B$ and $B \leqslant A$;
(3) $A^c = [\overline{A}^c, \underline{A}^c]$.

Definition 2 [1]. An intuitionistic fuzzy set in X is an expression A given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\},$$

where

$$\mu_A : X \to [0,1], \quad \nu_A : X \to [0,1],$$

with the condition $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$ for all $x \in X$. The set of all the IFSs in X is denoted by $IFSs(X); (a,b)_X$ is the IFS of X for which $(a,b)_X(x) = (a,b), \forall x \in X$.

The following expressions are defined in [1,4] for all $A, B \in IFSs(X)$:

(1) $A \leqslant B$ if and only if $\mu_A(x) \leqslant \mu_B(x)$ and $\nu_A(x) \geqslant \nu_B(x)$ for all $x \in X$;
(2) $A = B$ if and only if $A \leqslant B$ and $B \leqslant A$;
(3) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}$;
(4) $\pi_{A(x)} = 1 - \mu_A(x) - \nu_A(x)$ is called the intuitionistic index of the element x in A. It is evident that $0 \leqslant \pi_{A(x)} \leqslant 1$ for all x.

2.2. Relationship between IVFSs and IFSs

In 1989, Atanassov and Gargov [3] using a transformation from $[\underline{A}(x), \overline{A}(x)]$ to $(\underline{A}(x), 1 - \overline{A}(x)) = (\mu_A(x), \nu_B(x))$ proved that $IFSs(X)$ and $IVFSs(X)$ are equipollent generalizations of the notion of $FSs(X)$. Deschrijver and Kerre [8] proposed the general correspondence of IVFSs, IFSs and L-fuzzy set. Dubois et al. [9] discussed the terminological difficulties in fuzzy set theory and preferred the terminologies "interval-valued". Although $IFSs(X)$ and $IVFSs(X)$ have arisen on different grounds and have different semantics, sometimes they are even mathematically equivalent. So these approaches are in general not independent and there exists a strong connection between interval-valued fuzzy sets and intuitionistic fuzzy sets.

Taking this fact into account, all the definitions, propositions and theorems we introduce for the $IVFSs(X)$ henceforth, can be transformed to the case of $IFSs(X)$.

2.3. Distance and similarity measure for IVFSs

Liu [17], the prime author of the axiomatic definitions, offered axioms for fuzzy distance and similarity measure. As Liu [17] did, we now present the axioms that distance and similarity measure of IVFSs must satisfy.

Definition 3 [17]. A real function $D : IVFSs(X) \times IVFSs(X) \to [0,1]$ is called a distance, if D has the following properties:

(DP1) $D(A,B) = D(B,A)$, for all $A, B \in IVFSs(X)$;
(DP2) $D(A,A^c) = 1 \Longleftrightarrow A \in \mathscr{P}(X)$;
(DP3) $D(A,B) = 0 \Longleftrightarrow A = B$, for all $A, B \in IVFSs(X)$;
(DP4) for all $A, B, C \in IVFSs(X)$, if $A \leqslant B \leqslant C$, then $D(A,B) \leqslant D(A,C)$ and $D(B,C) \leqslant D(A,C)$.

Definition 4 [17]. A real function $S : IVFSs(X) \times IVFSs(X) \to [0,1]$ is called a similarity measure between IVFSs, if S has the following properties:

(SP1) $S(A,B) = S(B,A), \forall A, B \in IVFSs(X)$;
(SP2) $S(A,A^c) = 0 \Longleftrightarrow A \in \mathscr{P}(X)$;
(SP3) $S(A,B) = 1 \Longleftrightarrow A = B$;
(SP4) for all $A, B, C \in IVFSs(X)$, if $A \leqslant B \leqslant C$, then $S(A,C) \leqslant S(A,B)$ and $S(A,C) \leqslant S(B,C)$.

Atanassov [2] suggested a direct generalization of the Hamming distance and Euclidean distance used in classical set theory for intuitionistic fuzzy sets. Grzegorzewski [13] proposed new intuitionistic fuzzy distances between interval-valued fuzzy sets and/or intuitionistic fuzzy sets. Wang and Xin [30] gave a simplified axiom definition of distance measure between IFSs and proposed some intuitionistic fuzzy distance measures. Now we give the relevant distances for IVFSs according to the relationship between IVFSs and IFSs. For any $A, B \in IVFSs(X)$ with $\phi_{AB}(i) = |\underline{A}(x_i) - \underline{B}(x_i)|$ and $\varphi_{AB}(i) = |\overline{A}(x_i) - \overline{B}(x_i)|$, we get the following distances between two interval-valued fuzzy sets A and B.

- the normalized Euclidean distance:

$$D_1(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}[\phi_{AB}^2(i) + \varphi_{AB}^2(i)]}.$$

- the normalized Hamming distance:

$$D_2(A,B) = \frac{1}{2n}\sum_{i=1}^{n}[\phi_{AB}(i) + \varphi_{AB}(i)].$$

- the normalized Hamming distance based on Hausdorff metric: