Neural Networks 55 (2014) 98–109

Contents lists available at ScienceDirect

Neural Networks

journal homepage: www.elsevier.com/locate/neunet

Stability analysis of fractional-order Hopfield neural networks with time delays

Hu Wang, Yongguang Yu *, Guoguang Wen

Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, PR China

HIGHLIGHTS

- Stability of fractional-order time delay neural networks with hub structures is analyzed.
- Fractional-order neural networks with ring structures and time delays are developed.
- Stability conditions of the developed fractional-order neural networks are obtained.
- The influence of time delays on the stability is discussed.

ARTICLE INFO

Article history:
Received 10 June 2013
Received in revised form 17 February 2014
Accepted 30 March 2014
Available online 13 April 2014

Keywords:
Hopfield neural networks
Stability
Fractional-order
Time delays
Hub structure
Ring structure

ABSTRACT

This paper investigates the stability for fractional-order Hopfield neural networks with time delays. Firstly, the fractional-order Hopfield neural networks with hub structure and time delays are studied. Some sufficient conditions for stability of the systems are obtained. Next, two fractional-order Hopfield neural networks with different ring structures and time delays are developed. By studying the developed neural networks, the corresponding sufficient conditions for stability of the systems are also derived. It is shown that the stability conditions are independent of time delays. Finally, numerical simulations are given to illustrate the effectiveness of the theoretical results obtained in this paper.

1. Introduction

Since Hopfield's works in 1984 (Hopfield, 1984), Hopfield neural network has attracted great attention of many scientists, and it has been applied in various fields, such as pattern recognition, associative memory and combinatorial optimization (Carpenter, 1989; Cochocki & Rolf, 1993; Schliebs, Defoin-Platel, Worner, & Kasabov, 2009). In recent years, the researchers found that the fractional-order calculus could be well used in the study of Hopfield neural networks, due to the fact that the fractional-order differentiation provides neurons with a fundamental and general computational ability that contributes to efficient information processing, stimulus anticipation, and frequency-independent phase shifts in oscillatory neuronal firings (Lundstrom, Higgs, Spain, & Fairhall, 2008). In Kaslik and Sivasundaram (2011), the authors pointed out that the common capacitance from the continuous-time integer-order Hopfield neural network can be replaced by the fractance, giving birth to the so-called fractional-order Hopfield neural network model.

Due to the development of the fractional-order calculus, a lot of results on fractional-order neural networks have been obtained (Boroomand & Menhaj, 2009; Chen, Chai, Wu, Ma, & Zhai, 2013; Dominik, Grzegorz, & Andrzej, 2011; Gardner, 2009; Kaslik & Sivasundaram, 2012a, 2012b; Li, Peng, & Gao, 2013; Yu, Hu, & Jiang, 2012). In Boroomand and Menhaj (2009), the stability of fractional-order neural networks was fully investigated through an energy-like function analysis. In Dominik et al. (2011) and Gardner (2009), the discrete time fractional-order artificial neural networks were presented. In Yu et al. (2012), the α-stability and α-synchronization for fractional-order neural networks were investigated. In Li et al. (2013), the authors pointed out that the stability in Yu et al. (2012) was not α-stability but Mittag-Leffler stability. There are also several recent literatures discussing the topics including chaos and chaotic synchronization in fractional-order neural networks, which can be found in Arena, Fortuna, and Porto (2000), Çelik and Demir (2010), Huang, Zhao, Wang, and Li (2012) and Zhou, Li, and Zhu (2008). The stability as a very essential topic in fractional-order systems has attracted increasing interest in recent years. Some results have been obtained (Deng, Li,

* Corresponding author. Tel.: +86 13439090121.
E-mail addresses: wanghu1985712@163.com (H. Wang), ygyu@bjtu.edu.cn (Y. Yu), guoguang.wen@bjtu.edu.cn (G. Wen).

http://dx.doi.org/10.1016/j.neunet.2014.03.012
0893-6080/

& Lü, 2007; Hadi, Dumitru, & Jalil, 2012; Lazarević, 2011; Li, Chen, & Podlubny, 2009; Sabatier, Moze, & Farges, 2010). Hence, it is time to study the stability of fractional-order neural networks.

It is well known that the coordinated activation of neuronal assemblies is essential for the establishment of proper network wiring. Theoretically, a few highly connected neurons with long-ranging connectivity, called "hub neurons", would be the most efficient way to orchestrate network-wide synchronicity (Wiedemann, 2010). In scale-free networks, some nodes, which have been called "hubs", have many more connections than others, and the network as a whole has a power-law distribution of the number of links connecting to a node (Barabási & Albert, 1999). The existence of hub structures is a common feature, playing a fundamental role in defining the connectivity of the scale-free networks and in characterizing their dynamical behavior (Kitajima & Kurths, 2009). It is also well known that ring architectures have been found in a variety of neural structures, for example, in the hippocampus, cerebellum, neocortex, and even in chemistry and electrical engineering (Hirsch, 1989; Kaslik & Sivasundaram, 2012a). The real cortical connectivity pattern is extremely sparse. In fact, most connections are between nearby cells, and long-range connections are becoming more and more infrequent. Therefore, the ring represents a simplified connectivity structure, and ring neural networks can be studied to gain insight into the mechanisms underlying the behavior of recurrent networks (Hirsch, 1989). Recently, some results for different dynamical aspects of integer-order continuous-time neural networks with ring structures have been considered (Bungay & Campbell, 2007; Feng & Plamondon, 2012; Guo, 2005; Guo & Huang, 2007; Kaslik & Balint, 2009). From the above discussion, the hug and ring structures are common and basic in neural networks. By studying these simplified connectivity structures, the researchers could gain insight into the mechanisms underlying the behavior of complex recurrent networks, which will be helpful to investigate the more complicated neural networks (Kaslik & Sivasundaram, 2011, 2012a). Hence, it is worth investigating the stability of fractional-order neural networks with the hug structures or ring structures. In Kaslik and Sivasundaram (2012a), several topics, such as stability, multi-stability, bifurcations and chaos, related to the dynamics of fractional-order Hopfield neural networks with ring or hub structures but without time delay were investigated.

Note that most of the above results on the stability of fractional-order neural networks did not consider time delay. In fact, in practice because of finite switching speeds of the amplifiers, time delay is well-known to be unavoidable and can cause oscillations or instabilities in dynamic systems. Hence, it is very important to consider the stability of fractional-order neural networks with time delay. To the best of our knowledge, there are just few results on the stability of fractional-order neural networks with time delay. In Chen et al. (2013), a sufficient condition was established for the uniform stability of fractional-order neural networks with time delay. In Wang, Yu, Wen, and Zhang (submitted for publication), the fractional-order neural networks of two and three neurons with time delay were discussed, and the stability conditions were derived.

Motivated by the above discussion, this paper is devoted to presenting a theoretical stability analysis for fractional-order Hopfield neural networks with time delays. Firstly, some sufficient conditions are proposed for stability of the fractional-order Hopfield neural networks with hub structure and time delays. Especially, the multiple time delays are considered in the fractional-order Hopfield neural networks with hub structure. Secondly, to the best of our knowledge, there are few results on fractional-order neural networks with different ring structures. Then the fractional-order neural network with different ring structures as well as time delays are developed in this paper. By studying the developed neural networks, the corresponding sufficient conditions for stability of the systems are derived. In addition, this paper also considers the fractional-order Hopfield neural network with the ring structure time delay feedback. Finally, the influence of time delays on the stability is discussed. It is shown that the stability conditions are independent of time delays.

The paper is structured as follows. In Section 2, the preliminaries concerning fractional-order differential systems with time delays are introduced. In Section 3, the stability of fractional-order Hopfield neural networks with time delays is investigated, and the corresponding stability conditions are derived. In Section 4, the numerical simulations are given to illustrate the effectiveness of our theoretical results. Some conclusions are included in Section 5.

2. Preliminaries

In this section, the well-known results about the Caputo fractional-order derivative and the Adams–Bashforth–Moulton predictor–corrector scheme are briefly introduced. The main theoretical tools for the qualitative analysis of fractional-order dynamical systems are given in Kilbas, Srivastava, and Trujillo (2006) and Podlubny (1999). The Caputo derivative will be used in this paper.

The Caputo fractional-order derivatives are defined as follows:

$${}_0^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{q-n+1}} d\tau,$$

where n is an integer, $n-1 < q < n$ and $\Gamma(\cdot)$ is the Gamma function.

The Laplace transform of the Caputo fractional-order derivatives is

$$L\{{}_0^C D_t^q f(t);\ s\} = s^q F(s) - \sum_{k=0}^{n-1} s^{q-k-1} f^{(k)}(0), \quad n-1 < q \le n,$$

if $f^k(0) = 0$, $k = 1, 2, \ldots, n$, then

$$L\{{}_0^C D_t^q f(t);\ s\} = s^q F(s).$$

Properties of the Caputo fractional-order derivative include (Li & Deng, 2007):

(1) ${}_0^C D_t^q c = 0$ where c is any constant;
(2) If $x(t) \in C^m[0,T]$ for $T > 0$ and $m-1 < q < m \in Z^+$, then ${}_0^C D_t^q x(0) = 0$;
(3) If $x(t) \in C^1[0,T]$ for some $T > 0$, $q_1, q_2 \in R^+$, $q_1 + q_2 \le 1$, then ${}_0^C D_t^{q_1} {}_0^C D_t^{q_2} x(t) = {}_0^C D_t^{q_1+q_2} x(t)$.

To solve the differential equations of fractional-order with time delay (Bhalekar & Varsha, 2011), the Adams–Bashforth–Moulton predictor–corrector scheme will be used in this paper. Consider the following system

$$\begin{cases} {}_0^C D_t^q y(t) = f(t, y(t), y(t-\tau)), & t \in [0,T], 0 < q \le 1, \\ y(t) = g(t), & t \in [-\tau, 0], \end{cases}$$

$g(t)$ is a given initial function. And a uniform grid $t_n = nh : n = -k, -k+1, \ldots, -1, 0, 1, \ldots, N$, where k and N are integers such that $h = T/N$ and $h = \tau/k$.

Let

$$y_h(t_j) = g(t_j), \quad j = -k, -k+1, \ldots, -1, 0,$$

then it is easy to verify that

$$y_h(t_j - \tau) = y_h(t_j - kh) = y_h(t_{j-k}), \quad j = 0, 1, \ldots, N.$$

Suppose that the calculated approximations $y_h(t_j) \approx y(t_{j-k})$, $(j = -k, -k+1, \ldots, -1, 0, 1, \ldots, n)$ have been obtained.

Next, calculate $y_h(t_{n+1})$ using

$$y_h(t_{n+1}) = g(0) + \frac{1}{\Gamma(q)} \int_0^{t_{n+1}} (t_{n+1} - \zeta)^{q-1} f(\zeta, y(\zeta), y(\zeta - \tau)) d\zeta. \quad (1)$$