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# Synchronization of fractional-order complex-valued neural networks with time delay

Haibo Bao [a,b], Ju H. Park [b,*], Jinde Cao [c,d]

[a] School of Mathematics and Statistics, Southwest University, Chongqing 400715, PR China
[b] Nonlinear Dynamics Group, Department of Electrical Engineering, Yeungnam University, 280 Daehak-Ro, Kyongsan 38541, Republic of Korea
[c] Department of Mathematics, Southeast University, Nanjing 210096, PR China
[d] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

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ABSTRACT

This paper deals with the problem of synchronization of fractional-order complex-valued neural networks with time delays. By means of linear delay feedback control and a fractional-order inequality, sufficient conditions are obtained to guarantee the synchronization of the drive–response systems. Numerical simulations are provided to show the effectiveness of the obtained results.



## 1. Introduction

In recent years, complex-valued neural networks have gained a lot of attention because of their wide applications in electromagnetic, quantum waves, optoelectronics, filtering, speech synthesis, remote sensing, signal processing, and so on (Aizenberg, 2011; Aizenberg, Aizenberg, & Vandewalle, 2000; Amin & Murase, 2009; Cha & Kassam, 1995; Chen, Hanzo, & Tan, 2008; Hirose, 2012; Jankowski, Lozowski, & Zurada, 1996; Nitta, 2004; Tanaka & Aihara, 2009; Tripathi & Kalra, 2011).

Complex-valued neural networks are not only the simple extension of real-valued neural networks, but also are quite different from real-valued neural networks and have more complicated properties than real-valued neural networks. This is mainly because the state vectors, connection weights and activation functions in complex-valued neural networks are all complex values. Complex-valued neural networks can solve some problems that cannot be solved with their real-valued counterparts. For instance, the exclusion OR (XOR) problem and the detection of symmetry problem cannot be solved with a single complex-valued neuron with the orthogonal decision boundaries, which reveals the patent computational power of complex-valued neurons (Hirose, 1992).

Due to the finite switching speed of amplifiers, time delay inevitably exists in neural networks. It can cause oscillation and instability behavior of systems (Cao & Xiao, 2007; Lu, 2002; Wei & Ruan, 1999). Therefore, the study on stability of complex-valued neural networks with time delays is of both theoretical and practical importance. Up until now, there are some excellent results about the stability of complex-valued neural networks (Chen & Song, 2013; Gong, Liang, & Cao, 2015a, 2015b; Hu & Wang, 2015; Pan, Liu, & Xie, 2015; Rakkiyappan, Velmurugan, & Cao, 2015; Song, Zhao, & Liu, 2015a, 2015b; Velmurugan, Rakkiyappan, & Cao, 2015; Xu, Zhang, & Shi, 2014; Zhou & Song, 2013). The $\mu$-stability of complex-valued neural networks with unbounded time-varying delays was investigated in Gong et al. (2015b), Rakkiyappan, Velmurugan, and Cao (2015) and Velmurugan et al. (2015). In Song et al. (2015b), based on Lyapunov–Krasovskii functional and inequality technique, sufficient conditions are derived to guarantee the stability of complex-valued neural networks with probabilistic time-varying delays. In Zhou and Song (2013), the problem of boundedness and complete stability of complex-valued neural networks with time delay was investigated by using local inhibition and linear matrix inequalities method.

Fractional calculus dates from the 17th century and is the generation of integer-order calculus. Nowadays, many known systems can be described by fractional-order systems, such as viscoelasticity, dielectric polarization, and electromagnetic

* Corresponding author. Tel.: +82 53 810 2491; fax: +82 53 810 4767.
*E-mail addresses:* hbbao07@gmail.com (H. Bao), jessie@ynu.ac.kr (J.H. Park), jdcao@seu.edu.cn (J. Cao).

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waves. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and process (Kilbas, Srivastava, & Trujillo, 2006; Podlubny, 1999). The study of fractional-order differential equations has attracted the interest of many researchers in the fields of science and engineering (Raja, Khan, & Qureshi, 2010, 2011; Raja, Manzar, & Samar, 2015). So, it would be better if neural networks are described by fractional-order systems rather than integer-order ones. The dynamics analysis of fractional-order neural networks has been an active area of research (Chen, Zeng, & Jiang, 2014; Huang, Zhao, Wang, & Li, 2012; Kaslik & Sivasundaram, 2012; Tang, Wang, & Fang, 2009; Wu, Lu, & Chen, 2015; Yu, Hu, & Jiang, 2015; Yu, Hu, Jiang, & Fan, 2014). Huang et al. (2012) investigated chaos and hyperchaos of a fractional-order four-cell cellular neural network by means of numerical simulations. It is shown in Yu et al. (2015, 2014) that projective synchronization can be achieved by combining loop-open control and fractional-order inequality technique. In Chen et al. (2014), global Mittag-Leffler stability and synchronization of memristor-based fractional-order neural networks were investigated. Considering the effects of the time delays, many researchers have focused their attention on the stability and synchronization analysis of fractional-order neural networks and many interesting results have appeared, see Bao, Park, and Cao (2015), Chen, Chai, Wu, Ma, and Zhai (2013), Chen and Chen (2015), Wang, Yang, and Hu (2015), Wang, Yu, and Wen (2014) and Yang, Song, Liu, and Zhao (2015) and the references therein.

Although fractional-order systems broaden conventional integer-order systems, the stability and synchronization for fractional-order nonlinear systems are not well developed and still need further investigation because of higher complexity of fractional-order systems and absence of effective analytic tools. Till date, it is still difficult and even impossible for extending some properties of integer-order systems to fractional-order systems. For example, one can see how difficult it is to use quadratic Lyapunov functions to analyze the stability in Example 14 in Li, Chen, and Podlubny (2009), where the simple scalar fractional differential equation $D^{\alpha}x(t) = -x^3(t)$ is considered. It should be pointed out that most of the results about complex-valued neural networks are integer-order ones. There are only a few results investigating the dynamics of fractional-order complex-valued neural networks (Rakkiyappan, Cao, & Velmurugan, 2015; Rakkiyappan, Velmurugan, & Cao, 2014). It is well known that synchronization is one of the most important and interesting phenomenon of dynamical systems that exists in natural and man-made systems. There exist many benefits of having synchronization or chaos synchronization in some engineering applications, such as secure communication, image processing and harmonic oscillation generation. As far as the authors know, there are few results about the synchronization of the fractional-order complex-valued neural networks, which remains as an open challenge. So, it is necessary to investigate the synchronization of fractional-order complex-valued neural networks.

Motivated by the above discussions, the objective of this paper is to study the synchronization of fractional-order complex-valued neural networks. The main contributions of this paper can be summarized as follows: (1) A new delay feedback controller is designed to achieve the synchronization between the drive system and the response system, and this is the first time to investigate the synchronization of fractional-order complex-valued neural networks with time delay. (2) Compared with other results, the results of this paper are the ones about complex values and with time delay. Therefore, the results are less conservative and more general. (3) Some well-studied results (Yu et al., 2014) are the special cases of our results.

This paper is organized as follows. In Section 2, the problem description and preliminaries are presented. In Section 3, new criteria for synchronization between drive–response systems are derived. A numerical example is given to prove the main results of this paper in Section 4. Finally, the conclusions are drawn in Section 5.

*Notation:* The notations are very standard. Throughout this paper, $\mathcal{R}, \mathcal{C}^n, \mathcal{R}^{m\times n}$ and $\mathcal{C}^{m\times n}$ denote the set of real numbers, the $n$-dimensional complex vector space, the set of all $m \times n$ real and complex matrices, respectively. The subscripts $*$ and $T$ denote matrix complex conjugation and transposition and matrix transposition, respectively. Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}, a, b \in \mathcal{R}$. $|z| = \sqrt{a^2 + b^2}$, $\bar{z}$ denotes the conjugate complex number of $z$, $\bar{z} = a - ib$.

## 2. Model description and preliminaries

In this section, we will recall some definitions and lemmas which will be needed later. It is well known that there are several kinds of definitions of fractional integrals and derivatives, such as the Riemann–Liouville fractional integral and derivative and so on (Podlubny, 1999). In this paper, we will adopt Caputo fractional derivative.

**Definition 1** (*Kilbas et al., 2006; Podlubny, 1999*)**.** The fractional integral of order $\alpha$ for a function $f$ is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s)ds,$$

where $t \geq t_0$ and $\alpha > 0$.

**Definition 2** (*Kilbas et al., 2006; Podlubny, 1999*)**.** Caputo's derivative of order $\alpha$ for a function $f \in \mathcal{C}^n([t_0, +\infty), \mathcal{R})$ is defined by

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-s)^{n-\alpha-1}f^{(n)}(s)ds,$$

where $t \geq t_0$ and $n$ is a positive integer such that $n-1 < \alpha < n$.

Particularly, when $0 < \alpha < 1$, $D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}f'(s)ds$.

Consider a continuous fractional-order complex-valued neural network with time delay as the drive system described by

$$D^{\alpha}z(t) = -Cz(t) + Af(z(t)) + Bg(z(t-\tau)) + J(t), \quad (1)$$

where $0 < \alpha < 1$ denotes the order of fractional-order derivative, $z = (z_1, z_2, \ldots, z_n)^T \in \mathcal{C}^n$ is the state vector, $C = \text{diag}\{c_1, c_2, \ldots, c_n\} \in \mathcal{R}^{n\times n}$ is the self-feedback connection weight matrix with $c_j > 0$ $(j = 1, 2, \ldots, n)$, $\tau$ is the time delay, $A = (a_{jk})_{n\times n} \in \mathcal{C}^{n\times n}$ and $B = (b_{jk})_{n\times n} \in \mathcal{C}^{n\times n}$ are the connection weight matrix at time $t$ and time $t-\tau$. $f(z(t)) = (f_1(z_1(t)), f_2(z_2(t)), \ldots, f_n(z_n(t)))^T : \mathcal{C}^n \to \mathcal{C}^n$ and $g(z(t-\tau)) = (g_1(z_1(t-\tau)), g_2(z_2(t-\tau)), \ldots, g_n(z_n(t-\tau)))^T : \mathcal{C}^n \to \mathcal{C}^n$ are the complex-valued vector-valued activation functions at time $t$ and time $t-\tau$, and $J(t) = (J_1(t), J_2(t), \ldots, J_n(t))^T \in \mathcal{C}^n$ is the external input vector.

The response system is as follows

$$D^{\alpha}\tilde{z}(t) = -C\tilde{z}(t) + Af(\tilde{z}(t)) + Bg(\tilde{z}(t-\tau)) + J(t) + U(t). \quad (2)$$

The initial conditions associated with (1) and (2) are given by the following form:

$$z_j(s) = \phi_j(s) + i\psi_j(s),\ \tilde{z}_j(s) = \tilde{\phi}_j(s) + i\tilde{\psi}_j(s),\ s \in [-\tau, 0],$$

where $\phi_j(s), \psi_j(s), \tilde{\phi}_j(s)$, and $\tilde{\psi}_j(s) \in C([-\tau, 0], \mathcal{R}^n), j = 1, 2, \ldots, n$.

It is known that the activation functions in real-valued neural networks are usually chosen to be smooth and bounded. But according to Liouville's Theorem (Mathews & Howell, 2012), every bounded entire function in complex domain must be constant.