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Global exponential periodicity and stability of discrete-time complex-valued recurrent neural networks with time-delays☆

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ABSTRACT

In recent years, complex-valued recurrent neural networks have been developed and analysed in-depth in view of that they have good modelling performance for some applications involving complex-valued elements. In implementing continuous-time dynamical systems for simulation or computational purposes, it is quite necessary to utilize a discrete-time model which is an analogue of the continuous-time system. In this paper, we analyse a discrete-time complex-valued recurrent neural network model and obtain the sufficient conditions on its global exponential periodicity and exponential stability. Simulation results of several numerical examples are delineated to illustrate the theoretical results and an application on associative memory is also given.

1. Introduction

In the recent decades, complex-valued neural networks (CVNNs for short) that can directly handle complex-valued elements have been developed and analysed in-depth.

CVNNs, as an extension of real-valued recurrent neural networks, have complex-valued states, connections weights and activation functions. In many engineering applications, such as adaptive signal processing, communication engineering and medical imagine, the inputs and outputs of a system are presented as complex-valued elements. In such cases, CVNNs are well suited to handle these applications. Besides, CVNNs can solve some problems that cannot be solved by real-valued recurrent neural networks, see Nitta (2003). In the past few years, the study of CVNNs is a fast growing area, including learning algorithm, engineering optimization, image processing, pattern recognition and so on, see Aizenberg and Moraga (2007), Aizenberg, Moraga, and Paliy (2005), Aizenberg, Paliy, Zurada, and Astola (2008), Bohner, Rao, and Sanyal (2011), Duan and Song (2010), Goh and Mandic (2004, 2007), Hirose (2003, 2006, 2010), Hu and Wang (2012), Jankowski, Lozowski, and Zurada (1996), Kobayashi (2010), Lee (2001a, 2001b, 2006), Li, Liao, and Yu (2002), Liu, Fang, and Liu (2009), Nitta (2009), Rao and Murthy (2008), Rakkiyappan, Velmurugan, and Cao (2015), Zhou and Zurada (2009) and the references herein.

The dynamics of recurrent neural networks are frequently studied in recent years, see Cao and Wang (2003), Cao and Chen (2004), Cao and Wang (2005a, 2005b). When implementing the continuous-time neural networks for simulation, experimentation or computation, it is essential to construct a discrete-time neural network which is an analogue of the continuous-time neural network. Some researchers have discussed the significance for discrete-time analogues to reflect the dynamics of their continuous-time counterparts Mohamad and Naim (2002), Stuart and Humphries (1996). It is usually expected that the dynamical characteristics of the continuous-time systems pass to their discrete-time analogues. Although there are numerous ways of obtaining discrete-time neural networks from continuous-time neural networks, most of the discrete-time neural networks do not faithfully preserve the dynamics of their continuous-time counterparts. As pointed out in Hu and Wang (2006), the discretization cannot preserve the dynamics of the continuous-time counterpart even for a small sampling period, and therefore there is a crucial need to study the dynamics of discrete-time neural networks.

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A commonly used method in the formulation of a discrete-time analogue is discretizing the continuous-time system, see Mohamad and Gopalsamy (2000), Sun and Feng (2004), Yan (2011). Certainly, the discrete-time analogue when derived as a numerical approximation of continuous-time system is desired to preserve the dynamical characteristics of the continuous-time system. Once this is established, the discrete-time analogue can be used without loss of functional similarity to the continuous-time system and preserving any physical or biological reality that the continuous-time system has. The analysis and application of recurrent neural networks rely crucially on the stability and periodicity of the neural networks. Thus it is necessary and significant to study the dynamical behaviours of CVNNs. In recent years, the dynamical behaviours of some discrete-time CVNNs have been analysed, see Bohner et al. (2011), Du, Li, and Xu (2013), Duan and Song (2010), Hu and Wang (2002), Huang and Zhang (2010), Liu et al. (2009), Rao and Murthy (2008), Wang, Lu, and Chen (2009), Zhang, Yi, Zhang, and Heng (2009) and Zhou and Zurada (2009). However, the discrete-time analogues of continuous-time CVNNs are rarely studied.

In this paper, we use the semi-discretization technique to obtain discrete-time analogues of continuous-time CVNNs and study their global stability and exponential periodicity. The rest of the paper is organized as follows: In Section 2, we give the model description and some useful notations, definitions and lemmas. In Section 3, we present the sufficient condition for global exponential periodicity of discrete-time CVNNs with two types of activation functions and, as a result, we obtain the sufficient condition for global exponential stability as a special case. A numerical example is given in Section 4 to demonstrate the results. In this section, we also present a discrete-time CVNN to memorize the given patterns. Conclusion is given in Section 5.

2. Preliminaries

2.1. Notations

Let $x \in R$ be a real number, then $|x|$ denotes the absolute value of x. Let $z = x + \mathbf{i}y \in C$ be a complex number, where $\mathbf{i}$ denotes the imaginary unit, that is, $\mathbf{i} = \sqrt{-1}$, $x, y \in R$ be the real and imaginary part of z, then $|z|$ denotes the module of z, that is, $|z| = \sqrt{x^2 + y^2}$. z^* denotes the conjugate of z, that is, $z^* = x - \mathbf{i}y$.

Let $z = (z_1, z_2, \ldots, z_n) \in R^n$ be an n-dimensional complex-valued vector, where $z_i = x_i + \mathbf{i}y_i$ $(i = 1, 2, \ldots, n)$. z^* denotes the conjugate transpose of z, that is, $z = (z_1^*, z_2^*, \ldots, z_n^*)^{\mathrm{T}}$. The vector norm $\|z\|_1$ and $\|z\|_2$ (simply denoted by $\|z\|$) are defined as

$$\|z\|_1 = \sum_{i=1}(|x_i| + |y_i|), \qquad \|z\|_2 = \sum_{i=1}|z_i|. \tag{1}$$

For completeness we introduce the initial function space, $C([t_0 - \tau, t_0], \Omega)$, the Banach space of continuous functions $\phi : [t_0 - \tau, t_0] \to \Omega \subset R^n$ (C^n) with norm defined by

$$\|\phi\|_{t_0} = \sup_{-\tau \le \theta \le 0} \|\phi(t_0 + \theta)\|.$$

Let $A = (a_{ij})_{n\times n}$ be a complex-valued $n \times n$ matrix. A^{T} denotes the transpose of A. $|A|$ denotes a matrix composed by the module of each element of A, that is, $|A| = (|a_{ij}|)_{n\times n}$. As a special case, if $z = (z_1, z_2, \ldots, z_n)^{\mathrm{T}}$ is an n-dimensional complex-value vector, then $|z| = (|z_1|, |z_2|, \cdots, |z_n|)^{\mathrm{T}}$. If A and z are degenerated to real-valued matrix and vector, respectively, we can define $|A|$ and $|z|$ similarly.

2.2. Model descriptions

In this paper, the discrete-time analogue of the following continuous-time CVNN model is investigated:

$$\dot{z}(t) = -Dz(t) + Af(z(t)) + Bg(z(t - \tau)) + u(t) \tag{2}$$

where $z = (z_1, z_2, \ldots, z_n)^{\mathrm{T}} \in C^n$ is the state vector, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in R^{n\times n}$ with $d_i > 0$ $(i = 1, 2, \ldots, n)$ is the self-feedback connection weight matrix, $A = (a_{ij})_{n\times n} \in C^{n\times n}$ and $B = (b_{ij})_{n\times n} \in C^{n\times n}$ are, respectively, the connection weight matrix without and with time delays, $f(z(t)) = (f_1(z_1(t)), f_2(z_2(t)), \ldots, f_n(z_n(t)))^{\mathrm{T}} : C^n \to C^n$ and $g(z(t - \tau)) = (g_1(z_1(t - \tau_1)), g_2(z_2(t - \tau_2)), \ldots, g_n(z_n(t - \tau_n)))^{\mathrm{T}} : C^n \to C^n$ are the vector-valued activation functions without and with time delays, respectively, whose elements consist of complex-valued nonlinear functions, τ_i $(i = 1, 2, \ldots, n)$ are constant time delays, $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^{\mathrm{T}} \in C^n$ is the external input vector-valued function with period ω.

In this paper, we consider two types of complex-valued activation functions satisfying the following two assumptions:

Assumption 1. $f_i(z)$ $(i = 1, 2, \ldots, n)$ can be represented by separating into its real and imaginary part as:

$$f_i(z) = f_i^R(x, y) + \mathbf{i}f_i^I(x, y)$$

where $f_i^R(\cdot, \cdot) : R^2 \to R$ and $f_i^I(\cdot, \cdot) : R^2 \to R$. For $i = 1, 2, \ldots, n$, there exists positive numbers $\lambda_i^{RR}, \lambda_i^{RI}, \lambda_i^{IR}$ and λ_i^{II}, such that for any $x, x', y, y' \in R$ we have

$$\begin{aligned} |f_i^R(x, y) - f_i^R(x', y')| &\le \lambda_i^{RR}|x - x'| + \lambda_i^{RI}|y - y'|, \\ |f_i^I(x, y) - f_i^I(x', y')| &\le \lambda_i^{IR}|x - x'| + \lambda_i^{II}|y - y'|. \end{aligned} \tag{3}$$

Assumption 2. Let $f_i(\cdot)$ $(i = 1, 2, \ldots, n)$ be a set of complex-valued functions. For $i = 1, 2, \ldots, n$, there exists positive constant ξ_i such that for any $z, z' \in C$, we have

$$|f_i(z) - f_i(z')| \le \xi_i|z - z'|. \tag{4}$$

If activation functions $g_i(\cdot)$ satisfy Assumption 1, then there exists positive numbers μ_i^{RR}, μ_i^{RI}, μ_i^{IR} and μ_i^{II}, such that for any $x, x', y, y' \in R$ we have

$$\begin{aligned} |g_i^R(x, y) - g_i^R(x', y')| &\le \mu_i^{RR}|x - x'| + \mu_i^{RI}|y - y'|, \\ |g_i^I(x, y) - g_i^I(x', y')| &\le \mu_i^{IR}|x - x'| + \mu_i^{II}|y - y'|. \end{aligned} \tag{5}$$

If activation functions $g_i(\cdot)$ satisfy Assumption 2, then there exists positive constant κ_i such that for any $z, z' \in C$, we have

$$|g_i(z) - g_i(z')| \le \kappa_i|z - z'|. \tag{6}$$

By using a semi-discretization technique proposed in Mohamad and Gopalsamy (2000), we can obtain the discrete-time analogue of the continuous-time CVNN (2).

CVNN (2) can be represented by separating it into its real and imaginary part as:

$$\begin{aligned} \dot{x} &= -Dx + A^R f^R(x, y) - A^I f^I(x, y) \\ &\quad + B^R g^R(x^\tau, y^\tau) - B^I g^I(x^\tau, y^\tau) + u^R, \\ \dot{y} &= -Dy + A^I f^R(x, y) + A^R f^I(x, y) \\ &\quad + B^I g^R(x^\tau, y^\tau) + B^R g^I(x^\tau, y^\tau) + u^I. \end{aligned} \tag{7}$$

where $A^R = (a_{ij}^R)_{n\times n}$ and $A^I = (a_{ij}^I)_{n\times n}$ are, respectively, the real and imaginary part of A, $B^R = (b_{ij}^R)_{n\times n}$ and $B^I = (b_{ij}^I)_{n\times n}$ are, respectively, the real and imaginary part of B, $f^R(x, y) = (f_1^R(x_1, y_1), f_2^R(x_2, y_2), \ldots, f_n^R(x_n, y_n))^{\mathrm{T}}$ and $f^I(x, y) = (f_1^I(x_1, y_1),$