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Complete synchronization of temporal Boolean networks

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ABSTRACT

This letter studies complete synchronization of two temporal Boolean networks coupled in the drive-response configuration. Necessary and sufficient conditions are provided based on the algebraic representation of Boolean networks. Moreover, the upper bound to check the criterion is given. Finally, an illustrative example shows the efficiency of the proposed results.

1. Introduction

With the development of systems biology, genetic regulatory networks received much attention. Genetic regulatory networks can be modeled in quite different ways, such as differential equations (Shen, Wang, & Liang, 2011; Wang, Lam, Wei, Fraser, & Liu, 2008), Bayesian networks (Martínez-Rodríguez, May, & Vargas, 2008), and Boolean networks (BNs), (Kauffman, 1969).

Boolean network (BN) was first introduced by Kauffman (1969). In a Boolean network, 0 (or 1) corresponds to the inactive (or active) state of the gene. Every Boolean variable updates its state according to a Boolean function, which is a logical function. Since Boolean networks can provide a general description of the behavior of many living organisms at system level, they have attracted much attention in recent years. The topological structure of Boolean networks including attractors and transient time has been studied by Drossel, Mihaljev, and Greil (2005) and Samuelsson and Troein (2003), etc. Recently, the study has been extended to Boolean control networks, such as stability, controllability, observability, optimal control, etc. (Cheng & Qi, 2009; Cheng, Qi, Li, & Liu, 2011; Li & Sun, 2011, 2012; Li, Sun, & Wu, 2011; Li & Wang, 2012; Zhao, Li, & Cheng, 2011; Zhao, Qi, & Cheng, 2010).

The synchronization problem consists of making two systems oscillate in a synchronized manner, which can explain many natural phenomena. In the past decades, synchronization phenomena have been investigated experimentally, numerically and theoretically by many researchers; see e.g. Huang and Feng (2009), Shen, Wang, and Liu (2011, 2012), Su, Wang, and Lin (2009) and Zhu and Cui (2010). Recently, interest has extended to synchronization of Boolean networks (BNs) (Li & Chu, 2012; Li, Yang, & Chu, 2012), mostly due to their potential applications in biology, physics, etc. For example, the study of synchronized BNs could provide useful information on the coevolution of several biological species whose genetic dynamics influence each other (Morelli & Zanette, 2001).

The complete synchronization of two delay-free deterministic BNs has been studied in Li and Chu (2012). Later, the study was further extended in Li et al. (2012), yielding complete synchronization between two deterministic BNs with time delays coupled in the drive-response configuration. However, we can note that the time delays of every Boolean variables in Li et al. (2012) are the same. Compared with (Li et al., 2012), temporal Boolean network has more complex structure because temporal Boolean networks allow the time delays in each Boolean variable to be different. Hence, it is meaningful and challenging to investigate complete synchronization of two temporal Boolean networks. However, there has been no result studying the complete synchronization of two temporal Boolean networks, to the best of our knowledge.

Motivated by the above, in this letter, we focus on theoretical framework and strict analysis of complete synchronization between two temporal BNs coupled in the drive-response configuration. The main tool in this letter is the semi-tensor product of matrices, which can be used in many fields (Cheng, Qi, & Li, 2011; Wang, Zhang, & Liu, 2012). Based on the semi-tensor product of matrices, the algebraic representation of a temporal BN can be obtained. Then necessary and sufficient conditions for complete synchronization of two temporal BNs are obtained. Moreover, the upper bound to check the criterion is given.

The rest of the letter is organized as follows. Section 2 gives a brief review for the semi-tensor product of matrices. In Section 3, main results on synchronization of drive-response temporal BNs are presented. Necessary and sufficient conditions are obtained.

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Examples are given to show the non-emptiness of the obtained results in Section 4. Finally, Section 5 presents the conclusions.

2. Preliminaries

In this section, we give a brief introduction of the semi-tensor product (STP) and algebraic representation of Boolean functions which are proposed first by Daizhan Cheng and his colleagues (Cheng & Qi, 2010; Cheng, Qi, & Li, 2011).

2.1. Semi-tensor product of matrices

Definition 2.1 (*Cheng, Qi, & Li, 2011*)**.** For $M \in \mathbb{R}^{m\times n}$ and $N \in \mathbb{R}^{p\times q}$, their STP, denoted by $M \ltimes N$, is defined as follows:

$$M \ltimes N := (M \otimes I_{s/n})(N \otimes I_{s/p}),$$

where s is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

STP is a generalization of conventional matrix product. All the fundamental properties of conventional matrix product remain true. Based on this, we can omit the symbol "$\ltimes$" if no confusion raises. There are also some basic properties of STP; for detail, see Cheng and Qi (2010) and Cheng, Qi, and Li (2011).

2.2. Algebraic representation of Boolean functions

We need some notations first.

(1) $\mathcal{D} := \{1, 0\}$.
(2) $\Delta_k := \{\delta_k^i | i = 1, 2, \ldots, k\}$, where δ_k^i is the ith column of the identity matrix I_k.
(3) Denote by $\mathbf{1}_m$ a m-dimensional row vector whose entries are equal to 1.
(4) We denote the ith column of matrix A by $\mathrm{Col}_i(A)$ and denote the set of columns of matrix A by $\mathrm{Col}(A)$.
(5) Denote by $\mathcal{M}_{m\times n}$ the set of all $m \times n$ matrices. A matrix $A \in \mathcal{M}_{m\times n}$ is called a logical matrix, if the columns of A are elements of Δ_m. Denote the set of logical matrices by $\mathcal{L}$; the set of $n \times s$ logical matrices is defined by $\mathcal{L}_{n\times s}$.
(6) Assume that there is a matrix $M = [\delta_n^{i_1}, \delta_n^{i_2}, \ldots, \delta_n^{i_s}]$, for notational compactness, we simply denote it as $M = \delta_n[i_1, i_2, \ldots, i_s]$.

Second, by identifying $T = 1 \sim \delta_2^1, F = 0 \sim \delta_2^2$, then the logical variable $A(t)$ takes value from these two vectors, i.e. $A(t) \in \Delta := \Delta_2 = \{\delta_2^1, \delta_2^2\}$.

Next, some related results are collected in the following.

(1) A swap matrix, $W_{[m,n]}$, which is an $mn \times mn$ matrix constructed in the following way: label its columns by $(11, 12, \ldots, 1n, \ldots, m1, m2, \ldots, mn)$ and its rows by $(11, 21, \ldots, m1, \ldots, 1n, 2n, \ldots, mn)$. Then its element in the position $((I, J), (i, j))$ is assigned as

$$w_{(IJ),(ij)} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & \text{otherwise.} \end{cases}$$

When $m = n$, we briefly denote $W_{[n]} := W_{[n,n]}$. To see the properties of the swap matrix, we refer to Cheng and Qi (2010).

(2) The dummy matrix is defined as $E_d := \delta_2[1, 2, 1, 2]$. From Cheng and Qi (2010), we have for any two logical variables u, v,

$$E_d uv = v, \quad \text{or} \quad E_d W_{[2]} uv = u.$$

(3) Let Φ_n denote the $2^{2n} \times 2^n$ logical matrix

$$\Phi_n := \mathrm{Diag}\{\delta_{2^n}^1, \delta_{2^n}^2, \ldots, \delta_{2^n}^{2^n}\}$$

and let $\sigma \in \Delta_{2^n}$. Then $\sigma \times \sigma = \Phi_n \sigma$.

Finally, the following lemma is fundamental for the matrix expression of the logical function.

Lemma 2.1 (*Cheng, Qi, & Li, 2011*)**.** *Let $x_1, \ldots, x_n \in \mathcal{D}$ be n logical variables and $f(x_1, \ldots, x_n)$ a logical function. Then there exists a unique matrix $M_f \in \mathcal{L}_{2\times 2^n}$, called the structure matrix of f such that in the vector form, we have*

$$f(x_1, \ldots, x_n) = M_f \ltimes_{i=1}^n x_i, \quad x_i \in \Delta.$$

3. Main results

In this section, we will establish some necessary and sufficient conditions for complete synchronization between two temporal BNs.

3.1. Algebraic representation of temporal Boolean networks

In this letter, we consider two temporal BNs coupled in the drive-response configuration with each network consisting of n nodes. The dynamics can be described as

$$\begin{aligned} A_i(t+1) &= f_i(A_1(t), \ldots, A_n(t), A_1(t-1), \ldots, \\ &\qquad A_n(t-1), \ldots, A_1(t-\tau_1), \ldots, A_n(t-\tau_1)), \end{aligned} \tag{1}$$

$$\begin{aligned} B_i(t+1) &= g_i(A_1(t), \ldots, A_n(t), A_1(t-1), \ldots, \\ &\qquad A_n(t-1), \ldots, A_1(t-\tau_1), \ldots, A_n(t-\tau_1), \\ &\qquad B_1(t), \ldots, B_n(t), B_1(t-1), \ldots, \\ &\qquad B_n(t-1), \ldots, B_1(t-\tau_2), \ldots, B_n(t-\tau_2)), \end{aligned} \tag{2}$$

$i = 1, 2, \ldots, n,$

where A_i and B_i are the nodes of the drive BN (1) and the response BN (2) respectively; $f_i : \mathcal{D}^{n(\tau_1+1)} \to \mathcal{D}$, and $g_i : \mathcal{D}^{n(\tau_1+\tau_2+2)} \to \mathcal{D}$ are Boolean functions; and τ_1 and τ_2 are time delays, which are nonnegative integers.

In order to convert (1) into algebraic form, we define $A(t) = \ltimes_{i=1}^n A_i(t)$, $x(t) = \ltimes_{i=0}^{\tau_1} z_{ai}(t)$, where $z_{ai}(t) = A(t-i)$. Assume that the structure matrix of f_i and g_i are M_i and N_i respectively, we can express (1) as

$$A_i(t+1) = M_i A(t) A(t-1) \cdots A(t-\tau_1) = M_i x(t). \tag{3}$$

Multiplying the equations (3) together yields

$$\begin{aligned} A(t+1) &= M_1 x(t) M_2 x(t) \cdots M_n x(t) \\ &= L_1 x(t), \end{aligned} \tag{4}$$

where $\mathrm{Col}_i(L_1) = \ltimes_{j=1}^n \mathrm{Col}_i(M_j)$. Hence, the algebraic representation of the drive BN (1) is

$$\begin{aligned} x(t+1) &= \ltimes_{i=0}^{\tau_1} z_{ai}(t+1) = z_{a0}(t+1) \cdots z_{a\tau_1}(t+1) \\ &= A(t+1)A(t) \cdots A(t-\tau_1+1) \\ &= L_1 x(t) A(t) \cdots A(t-\tau_1+1) \\ &= L_1 W_{[2^{n\tau_1}, 2^{n(\tau_1+1)}]} \Phi_{n\tau_1} A(t) \cdots A(t-\tau_1+1) A(t-\tau_1) \\ &= L_1 W_{[2^{n\tau_1}, 2^{n(\tau_1+1)}]} \Phi_{n\tau_1} x(t) \\ &\triangleq Fx(t). \end{aligned}$$

Similarly, defining $B(t) = \ltimes_{i=1}^n B_i(t)$, $y(t) = \ltimes_{i=1}^{\tau_2} z_{bi}(t)$, where $z_{bi}(t) = B(t-i)$, we have

$$\begin{aligned} B_i(t+1) &= N_i A(t) A(t-1) \cdots A(t-\tau_1) \\ &\qquad B(t) B(t-1) \cdots B(t-\tau_2) \\ &= N_i x(t) y(t). \end{aligned} \tag{5}$$