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# Hodge–Kodaira decomposition of evolving neural networks

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ABSTRACT

Although it is very important to scrutinize recurrent structures of neural networks for elucidating brain functions, conventional methods often have difficulty in characterizing global loops within a network systematically. Here we applied the Hodge–Kodaira decomposition, a topological method, to an evolving neural network model in order to characterize its loop structure. By controlling a learning rule parametrically, we found that a model with an STDP-rule, which tends to form paths coincident with causal firing orders, had the most loops. Furthermore, by counting the number of global loops in the network, we detected the inhomogeneity inside the chaotic region, which is usually considered intractable.



## 1. Introduction

Although the network topology can be very important for communicating neurons, the conventional network analyses are often limited to locally defined variables such as degrees. Those intrinsically local variables cannot capture the global recurrent structures ubiquitously observed in neural networks. In fact, a network that alters its coupling strengths under STDP learning rule has tendency to make long paths of sequential firings (Aoki & Aoyagi, 2007, 2009, 2011, 2012; Buonomano, 2005; Edelman, Izhikevich, & Gally, 2004; Liu & Buonomano, 2009; Magnasco, Piro, & Cecchi, 2009; Masuda, Kawamura, & Kori, 2009; Morrison, Aertsen, & Diesmann, 2007; Takahashi, Kori, & Masuda, 2009; Tsubo, Teramae, & Fukai, 2007).

To characterize the global loop structures, the approaches based on algebraic topology are needed (Bossavit, 1997; Curto, Itskov, Veliz-Cuba, & Youngs, 2013; Fulton, 1995; Hatcher, 2002). Recent advances in the field of computational topology made it possible to compute topological invariants, such as the number of "holes" in proteins (Gameiro et al., 2012), in an accessible way (Arai, Kokubu, & Pilarczyk, 2009; Edelsbrunner & Harer, 2009; Kaczynski, Mischaikow, & Mrozek, 2010). For example, topological methods can count the number of "marbles" in an image irrespective of their shapes, which can be much more informative in detecting cancers than just using raw pixels. For discrete graphs, "graph invariants", independent of labeling, are desired (Chandrasekaran, Parrilo, & Willsky, 2012) in the same vein, especially when, say, you randomize initial conditions as in this paper and, therefore, do not care about specific labels.

Here we apply the Hodge–Kodaira decomposition of graph flows (de Rham, 1984; Hodge, 1941; Jiang, Lim, Yao, & Ye, 2011; Kodaira, 1949; Warner, 1983) to evolving neural network models (Aoki & Aoyagi, 2009) in order to count the number of global loops as a topological measure of network structures. Specifically, it is interesting to see if the measure reflects the bifurcation diagrams and even subdivides chaotic parameter regions, which are usually considered intractable.

In Section 2, we explain the evolving neural network model which we simulated. We also show the method of Hodge–Kodaira decomposition. In Section 3, we show the results of Hodge–Kodaira decomposition applied to the evolving neural networks. Finally, Section 4 presents a summary and discussions.

## 2. Materials and methods

### 2.1. Simulations

We simulated the following model of $N(=100)$ phase oscillators whose couplings evolve over time (Aoki & Aoyagi, 2009):

$$\frac{d\phi_i}{dt} = \omega - \frac{1}{N}\sum_{j=1}^{N} k_{ij}\sin(\phi_i - \phi_j + \alpha\pi)$$

$$\frac{dk_{ij}}{dt} = -\epsilon\sin(\phi_i - \phi_j + \beta\pi), \quad (1)$$

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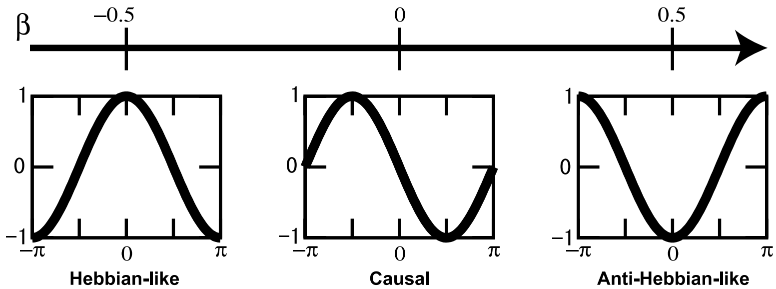


**Fig. 1.** Learning rule of the model: $\frac{dk_{ij}(\Delta\phi)}{dt}$ (see Eq. (1)). The parameter $\beta$ can control the learning rule.

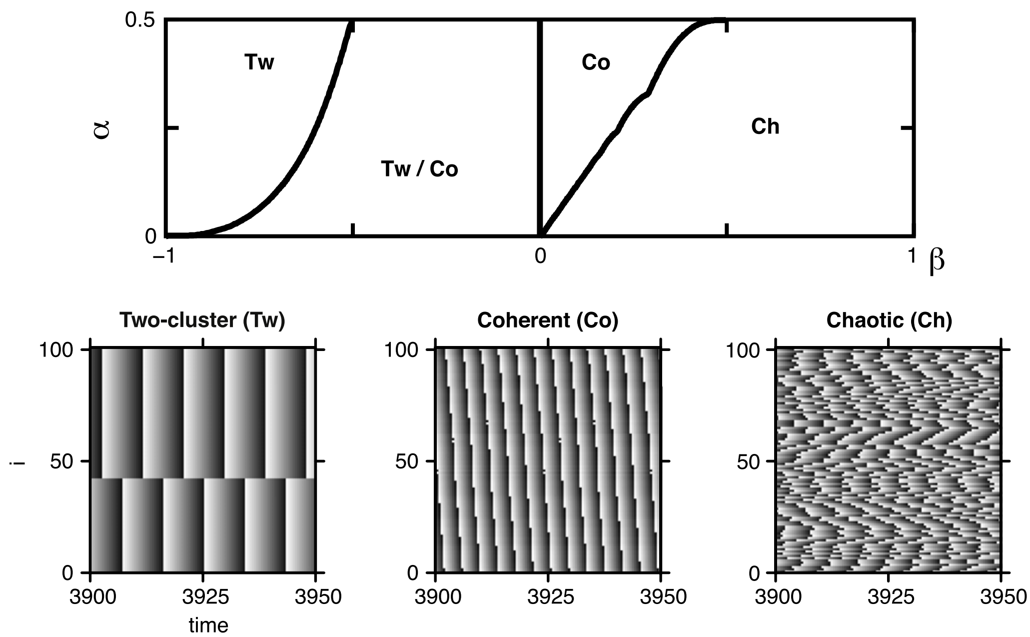


**Fig. 2.** Bifurcation diagram of the model consisting of three attractor states: two-cluster (Tw), coherent (Co) and chaotic (Ch) states (top). The dynamics of 100 neurons in each attractor state (bottom). The phase of each neuron in $[0, 2\pi]$ is represented by gradation. The parameter sets $(\alpha, \beta)$ are $(0.3, -0.6)$ (left), $(0.3, 0)$ (middle) and $(0.3, 0.6)$ (right). Note that although we simulated the network consisting of 100 neurons ($N = 100$), the phase diagram was analytically obtained for $N = \infty$ (Aoki & Aoyagi, 2011).

where $\phi_i$ and $k_{ij}$ denote the phase of $i$th neuron and the coupling strength from $j$th to $i$th neuron and we solely use $\omega = 1$ and $\epsilon = 0.005$. The learning scheme can be controlled by $\beta$: Hebb-rule for $\beta \sim -0.5$, STDP rule for $\beta \sim 0$, and anti-Hebb-rule for $\beta \sim 0.5$ (Figs. 1 and 2). We entirely used $\alpha = 0.3$, although the result did not change qualitatively when we used $\alpha = 0.1$.

The base, undirected and sparse network was randomly generated in the following way so that only limited coupling strengths ($k_{ij}$) can take non-zero (Bollobas, 2001). To construct an undirected graph, pairs of nodes were randomly connected with probability $p = 0.1$. We used $p = 0.1$ because the network with $p \gg 0.1$ does not have loops of lengths longer than 3 (i.e., no harmonic flow in Hodge–Kodaira decomposition) and the network with $p \ll 0.1$ gets disconnected (Kahle, 2009; Kahle & Meckes, 2013). Although we entirely used $p = 0.1$ in this paper, the result did not change qualitatively when we used $p = 0.05, 0.15, 0.2$ and 0.25. When two nodes of the base network are connected, both directions of couplings are allowed. As we avoid self-loops, the base network has 100 nodes and 1000 directed edges (=500 undirected edges).

The dimension of the (antisymmetric) "flow" was computed as the number of non-zero couplings after the time evolution where the initial coupling strengths and phases are randomized uniformly. Note that we interchangeably use flows and directed couplings in this paper. That is, the adjacency matrix consisting of the coupling strengths $K = \{k_{ij}\}$ is antisymmetrized as

$$A = \frac{K - K^t}{2}, \tag{2}$$

where $^t$ represents the transpose of a matrix. Then the dimension, that is, the number of non-zero components divided by two, of the antisymmetrized matrix, which represents the directed components of connections, is computed. We judged a coupling strength as non-zero when its absolute values are larger than the threshold (=0.05). The result did not change qualitatively when we used 0.2 for the threshold although we had smaller dimensions. Similarly, for the dimension of the symmetric flows, we computed the half number of non-zero component of the symmetrized matrix:

$$S = \frac{K + K^t}{2}. \tag{3}$$

Note that any matrix can be decomposed into the symmetric and antisymmetric matrices:

$$K = A + S. \tag{4}$$

In the next subsection, we show that the antisymmetric matrix can be further decomposed into three matrices uniquely by the Hodge–Kodaira decomposition.