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# Modified big-M method to recognize the infeasibility of linear programming models

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**Abstract**

This paper provides an effective modification to the big-M method which leads to reducing the iterations of this method, when it is used to recognize the infeasibility of linear systems.


*Keywords:* Linear programming; Simplex algorithm; Big-M method; Artificial variables

## 1. Introduction

The simplex algorithm was conceived by Dantzig for solving linear programming (LP) problems (see [1–3,7]). This method starts with a basic feasible solution (BFS) and moves to an improved BFS, until the optimal point is reached or else unboundedness of the objective function is verified. In order to initialize this algorithm a BFS must be available. In many cases, finding such a BFS is not straightforward and some work may be needed to get the simplex algorithm started. To this end, there are two techniques in linear programming literature: two-phase method and big-M method [1,2,6]. But there may be some LP models for which there are not any BFSs, i.e., the model is infeasible. Both two-phase method and big-M method distinguish the infeasibility. In this paper, we focus on infeasible cases and deal with the behaviour of big-M approach when dealing with infeasibility, and modify one of its end-conditions to strongly reduce the iterations required to distinguish the infeasibility.

The rest of this paper unfolds as follows: In Section 2, the big-M technique is reviewed. Section 3 contains the provided modification and in Section 4, to illustrate the ability of the provided modification to reduce the number of iterations, a class of LP models is surveyed. Finally, Section 5 contains a short conclusion.

## 2. Big-M method

Consider a generic LP model. After manipulating the constraints and introducing the required slack variables, the constraints are put in the format $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}$ where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b} \geqslant \mathbf{0}$ is an $m \times 1$ vector. Considering $\mathbf{c}$ as cost vector, the following LP model is dealt with:

$$\begin{aligned} \min\quad & \mathbf{cx} \qquad (P) \\ \text{s.t.}\quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}. \end{aligned}$$

Furthermore, suppose that we do not have a starting BFS for simplex method, i.e., $\mathbf{A}$ has no identity submatrix. In this case we shall resort to the artificial variables to get a starting BFS, and then use the simplex method itself and get rid of these artificial variables. The use of artificial variables to obtain a starting BFS was first provided by Dantzig[2].

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To illustrate, suppose that we change the restrictions by adding an artificial vector $\mathbf{x}_a$ leading to the system $\mathbf{Ax} + \mathbf{x}_a = \mathbf{b}, (\mathbf{x}, \mathbf{x}_a) \geqslant \mathbf{0}$. This forces an identity submatrix corresponding to the artificial vector and gives an immediate BFS of the new system, namely $(\mathbf{x} = \mathbf{0}, \mathbf{x}_a = \mathbf{b})$. Even though we now have a starting BFS and the simplex method can be applied, we have in effect changed the problem. In order to get back to our original problem, we must force these artificial variables to zero, because $\mathbf{Ax} = \mathbf{b} \Longleftrightarrow \mathbf{Ax} + \mathbf{x}_a = \mathbf{b}, \mathbf{x}_a = \mathbf{0}$.

There are various methods that can be used to eliminate the artificial variables. One of these methods is to assign a large penalty coefficient to these variables in the original objective function in such a way as to make their presence in the basis at a positive level very unattractive from the objective function point of view. More specifically, $(P)$ is changed to:

$$\begin{aligned} \min \quad & \mathbf{cx} + M\mathbf{1x}_a \qquad P(M) \\ \text{s.t.} \quad & \mathbf{Ax} + \mathbf{x}_a = \mathbf{b} \\ & (\mathbf{x}, \mathbf{x}_a) \geqslant \mathbf{0}, \end{aligned}$$

where $M$ is a very large positive number and $\mathbf{1} = (1, 1, \ldots, 1)$. The term $M\mathbf{1x}_a$ can be interpreted as a penalty to be paid by any solution with $\mathbf{x}_a \neq \mathbf{0}$. Therefore the simplex method itself will try to get the artificial variables out of the basis, and then continue to find an optimal solution of the original problem. This technique is named the big-M method. Hereafter "*" indicates the optimality, $z_j - c_j$ is the reduced cost of the $j$th variable, and $\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j$, where $\mathbf{B}$ is the basis of the simplex method associated with the related iteration and $\mathbf{a}_j$ is the $j$th column of the technological coefficients matrix (for $P(M)$ it is $[\mathbf{A,I}]$).

Four possible cases may arise while solving $P(M)$:

$(A_1)$ : $(\mathbf{x}^*, \mathbf{x}_a^*)$ is an optimal solution of $P(M)$, in which $\mathbf{x}_a^* = \mathbf{0}$.
$(A_2)$ : $(\mathbf{x}^*, \mathbf{x}_a^*)$ is an optimal solution of $P(M)$, in which $\mathbf{x}_a^* \neq \mathbf{0}$.
$(B_1)$ : $z_k - c_k = \max(z_j - c_j) > 0, \mathbf{y}_k \leqslant \mathbf{0}$, and all artificials are equal to zero.
$(B_2)$ : $z_k - c_k = \max(z_j - c_j) > 0, \mathbf{y}_k \leqslant \mathbf{0}$, and not all artificials are equal to zero.

In case $(A1)$; $\mathbf{x}^*$ is an optimal solution of $(P)$ and in case $(B1)$; $(P)$ has an unbounded optimal value. In this paper we focus on the two other cases, i.e., $(A_2)$ and $(B_2)$. The following theorem clarifies these cases. See pages 155–159 of [1] for details.

**Theorem 1.** *Cases* $(A2)$ and $(B2)$ *imply the infeasibility of* $(P)$.

Considering the above theorem and studying conditions $(A2)$ and $(B2)$, show that the big-M method recognizes the infeasibility of $(P)$ after completely solving $P(M)$ and this can be onerous from a computational point of view. In the next section, we modify the condition of case $(B2)$ to reduce this problem.

## 3. Modification of big-M method

In this section, we reduce the assumption of Theorem 1 and provide a reduced condition that leads to decreasing the iterations of the big-M method. The following theorem contains the modified condition.

**Theorem 2.** *Suppose that while solving $P(M)$, we have*

$$z_k - c_k = \max(z_j - c_j) > 0, \sum_{i \in \mathcal{B}_1} y_{ik} \leqslant 0 \tag{1}$$

*where*

$\mathcal{B}_1 = \{i | x_i \text{ is a basic artificial variable}\}$,

*and not all artificials are equal to zero. Then $(P)$ is infeasible.*

**Proof.** Suppose that sets $\mathcal{B}_1, \mathcal{B}_2$, and $\mathcal{N}$ consist of the indices of basic artificial variables, basic original variables, and nonbasic variables, respectively. $\mathbf{B}$ is the basis corresponding to the simplex tableau satisfying (1), $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$, and $\mathbf{c} = (\mathbf{c_B}, \mathbf{c_N})$. By these considerations we get

$$z_j - c_j = \mathbf{c_B y}_j - c_j = \sum_{i \in \mathcal{B}_2} c_i y_{ij} + M\left(\sum_{i \in \mathcal{B}_1} y_{ij}\right) - c_j.$$

Since $z_k - c_k = \max(z_j - c_j), \sum_{i \in \mathcal{B}_1} y_{ik} \leqslant 0, M$ is a very large number, and regarding the above equation we get $\sum_{i \in \mathcal{B}_1} y_{ij} \leqslant 0$ for all $j \in \mathcal{N}$.

Now by contradiction suppose that $(P)$ has a feasible solution, then $x_i = 0$, for all $i \in \mathcal{B}_1$. From the corresponding tableau we have

$$x_i + \sum_{j \in \mathcal{N}} x_j y_{ij} = \bar{b}_i \quad \text{for all } i \in \mathcal{B}_1.$$

Summing these equations gives

$$\sum_{j \in \mathcal{N}} x_j \left(\sum_{i \in \mathcal{B}_1} y_{ij}\right) = \sum_{i \in \mathcal{B}_1} \bar{b}_i.$$

The left-hand-side of this equation is $\leqslant 0$ while the other side is $> 0$. This contradiction shows that $(P)$ is infeasible and completes the proof. □

Note that condition (1) in Theorem 2 is a reduced version of that in case $(B2)$ in the previous section, i.e., it may happen that in one of the iterations of the big-M method, condition (1) holds while the condition of case $(B2)$ does not hold.

**Theorem 3.** *The condition provided in Theorem 2 is able to distinguish the infeasibility of $(P)$ before the condition of case $(B2)$ holds.*

**Proof.** Clearly we have

$$\begin{aligned} z_k - c_k = \max(z_j - c_j) > 0, \quad & \mathbf{y}_k \leqslant \mathbf{0} \Rightarrow z_k - c_k \\ = \max(z_j - c_j) > 0, \quad & \sum_{i \in \mathcal{B}_1} y_{ik} \leqslant 0 \end{aligned}$$