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# Impulsive hybrid discrete-time Hopfield neural networks with delays and multistability analysis

Eva Kaslik [a,b], Seenith Sivasundaram [c,*]

[a] Institute eAustria Timisoara, Bd. V. Parvan nr. 4, room 045B, 300223, Timisoara, Romania

[b] Department of Mathematics and Computer Science, West University of Timisoara, Bd. V. Parvan nr. 4, 300223, Romania

[c] Department of Mathematics, Embry-Riddle Aeronautical University, Daytona Beach, FL 32114, USA

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ABSTRACT

In this paper we investigate multistability of discrete-time Hopfield-type neural networks with distributed delays and impulses, by using Lyapunov functionals, stability theory and control by impulses. Example and simulation results are given to illustrate the effectiveness of the results.



## 1. Introduction

In recent years, there has been increasing interest in neural networks such as (Chua & Yang, 1988; Cohen & Grossberg, 1983; Hopfield, 1982), and bidirectional associative memory (Kosko, 1988) neural networks, and their potential applications in many areas such as classification, optimization, signal and image processing, solving nonlinear algebraic equations, pattern recognition, associative memories, cryptography and so on.

The state of electronic networks is often subject to instantaneous changes, and will experience abrupt changes at certain instants which can be caused by frequency change, switching phenomenon, or by some noise which do exhibit impulse effects.

In the past decades, a number of research papers have dealt with dynamical systems with impulse effect as a class of general hybrid systems. Examples include the pulse frequency modulation, optimization of drug distribution in the human body and control systems with changing reference signal. Impulsive dynamical systems are characterized by the occurrence of abrupt change in the state of the system which occur at certain time instants over a period of negligible duration. The dynamical behavior of such systems is much more complex than the behavior of dynamical systems without impulse effects. The presence of impulse means that the state trajectory does not preserve the basic properties which are associated with non-impulsive dynamical systems. Thus, the theory of impulsive differential equations is quite interesting and has attracted the attention of many scientists.

In general, most neural networks have been assumed to be in continuous time. Discrete-time counterparts of continuous-type neural networks have only been in the spotlight since 2000, even though they are essential when implementing continuous-time neural networks for practical problems such as image processing, pattern recognition and computer simulation. Discrete-time systems with delays have strong background in engineering applications, among which network based control has been well recognized to be a typical example. Discrete-time neural networks are more applicable to problems that are inherently temporal in nature or related to biological realities. They perfectly can keep the dynamic characteristics, functional similarity, and even the biological or physical resemblance of the continuous-time networks under certain mild conditions (restrictions) (Huo & Li, 2009; Mohamad, 2001, 2003, 2008; Mohamad & Gopalsamy, 2000). For this reason, the stability analysis of discrete-time neural networks have received more and more attention recently.

In the following, we use the notations

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}, \qquad \mathbb{Z}_0^+ = \{0, 1, 2, \ldots\},$$
$$\mathbb{Z}_0^- = \{\ldots, -2, -1, 0\}.$$

* Corresponding author.
*E-mail addresses:* ekaslik@gmail.com (E. Kaslik), seenithi@gmail.com (S. Sivasundaram).

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Continuous-time impulsive hybrid discrete Hopfield-type neural networks with distributed delays are described by

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i x_i(t) + \sum_{j=1}^{n} T_{ij} g_j\left(\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)\mathrm{d}s\right) + I_i, \quad t>0,\ t\neq t_k;\ i=\overline{1,n} \tag{1}$$
$$\mathbf{x}(t_k^-) = \mathbf{x}(t_k), \qquad \mathbf{x}(t_k^+) = \mathbf{x}(t_k) + J_k(\mathbf{x}), \quad t=t_k,\ k\in\mathbb{Z}^+$$

where the sequence of times $\{t_k\}_{k\in\mathbb{Z}_0^+}$ satisfies $0=t_0<t_1<t_2<\cdots<\lim_{k\to\infty} t_k=\infty$.

System is equivalent to

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -a_i x_i(t) + \sum_{j=1}^{n} T_{ij} g_j\left(\int_{0}^{\infty} K_{ij}(s)x_j(t-s)\mathrm{d}s\right) + I_i, \quad t>0,\ t\neq t_k;\ i=\overline{1,n} \tag{2}$$
$$\mathbf{x}(t_k^-) = \mathbf{x}(t_k), \qquad \mathbf{x}(t_k^+) = \mathbf{x}(t_k) + J_k(\mathbf{x}), \quad t=t_k,\ k\in\mathbb{Z}^+.$$

The constants $a_i > 0$ are the self-regulating parameters of the neurons, $T=(T_{ij})_{n\times n}$ is the interconnection matrix, $g_i:\mathbb{R}\to\mathbb{R}$ are the neuron input–output activation functions and $I_i$ denotes the external inputs.

The delay kernels $K_{ij}:[0,\infty)\to[0,\infty)$ are bounded, piecewise continuous and satisfy

$$\int_0^\infty K_{ij}(s)\mathrm{d}s = 1 \quad\text{and}\quad \exists\mu>0\quad\text{s.t.}\ \int_0^\infty K_{ij}(s)\mathrm{e}^{\mu s}\mathrm{d}s<\infty.$$

The jump operators $J_k$ are defined on the following set of functions:

$$\{\mathbf{u}\in PC((-\infty,t_k],\mathbb{R}^n): \mathbf{u}\text{ is left continuous, with first kind discontinuity at } t_l,\quad 0\le l\le k;\ \mathbf{u}\text{ is differentiable on every interval } (t_{l-1},t_l),\ 0\le l\le k\}$$

with values in $\mathbb{R}^n$.

The discrete analogue of system (2) is obtained in the following way.

Consider a positive number $h$ denoting a uniform discretization step size and $[t/h]$ the greatest integer in $t/h$. For convenience, we denote $[t/h]=p,\ p\in\mathbb{Z}$. We also note that $x_i(t)$ takes the form $x_i(ph)$, for $t\in[ph,(p+1)h)$. We shall use this approximation only for integers $p$ such that the interval $[ph,(p+1)h)$ contains no moment of impulse effect $t_k,\ k\in\mathbb{Z}^+$. That is, we assume that there is not more than one moment of impulse effect in a step. For this, we suppose that $\omega=\inf_{k\in Z^+}(t_{k+1}-t_k)>h>0$ and we denote $[t_k/h]=p_k,\ k\in\mathbb{Z}^+$. For simplicity, we will denote $x_i(ph)\equiv x_i(p)$ and $K_{ij}(vh)\equiv K_{ij}(v)$.

We rewrite system (2) in the form

$$\frac{\mathrm{d}[x_i(t)\mathrm{e}^{a_i t}]}{\mathrm{d}t} = \mathrm{e}^{a_i t}\left(\sum_{j=1}^{n} T_{ij} g_j\left(\sum_{v=1}^{\infty} K_{ij}(v)x_j(p-v)\right)+I_i\right), \quad t\in[ph,(p+1)h);\ i=\overline{1,n},\ p\in\mathbb{Z}_0^+\setminus\{p_1,p_2,\ldots\};$$
$$\mathbf{x}(p_k+1)=\mathbf{x}(p_k)+J_k(\mathbf{x}),\quad p=p_k,k\in\mathbb{Z}^+.$$

Now by integrating the first equation over the interval $[ph,t]$, for $t<(p+1)h$, we obtain

$$x_i(t)\mathrm{e}^{a_i t}-x_i(p)\mathrm{e}^{a_i ph} = \frac{\mathrm{e}^{a_i t}-\mathrm{e}^{a_i ph}}{a_i}\left(\sum_{j=1}^{n} T_{ij} g_j\left(\sum_{v=1}^{\infty} K_{ij}(v)x_j(p-v)\right)+I_i\right)$$
$$t\in[ph,(p+1)h);\ i=\overline{1,n},\ p\in\mathbb{Z}_0^+\setminus\{p_1,p_2,\ldots\}.$$

By letting $t\to(p+1)h$ we get the following discrete analogue system

$$x_i(p+1) = \mathrm{e}^{-a_i h}x_i(p) + \frac{1-\mathrm{e}^{-a_i h}}{a_i}\times\left(\sum_{j=1}^{n} T_{ij} g_j\left(\sum_{v=1}^{\infty} K_{ij}(v)x_j(p-v)\right)+I_i\right), \quad p\in\mathbb{Z}_0^+\setminus\{p_1,p_2,\ldots\};\ i=\overline{1,n} \tag{3}$$
$$\mathbf{x}(p_k+1)=\mathbf{x}(p_k)+J_k(\mathbf{x}),\quad p=p_k,\ k\in\mathbb{Z}^+.$$

If we set $\Psi_i(h)=\frac{1-\mathrm{e}^{-a_i h}}{a_i}$, for $i=\overline{1,n}$, it is easy to see that $\Psi_i(h)>0$. It is clear that the equilibria of continuous-time system (2) and discrete-time analogue (3) coincide.

In this paper, we will be studying the more general discrete-time impulsive system with distributed delays of the form

$$x_i(p+1) = (1-a_i)x_i(p) + \sum_{j=1}^{n} T_{ij} g_j\left(\sum_{v=1}^{\infty} K_{ij}(v)x_j(p-v)\right)+I_i, \quad p\in\mathbb{Z}_0^+\setminus\{p_1,p_2,\ldots\};\ i=\overline{1,n} \tag{4}$$
$$\mathbf{x}(p_k+1)=\mathbf{x}(p_k)+J_k(\mathbf{x}),\quad p=p_k,\ k\in\mathbb{Z}^+$$

where $a_i\in(0,1)$ and the sequence of times $\{p_k\}_{k\in\mathbb{Z}_0^+}$ satisfies $0=p_0<p_1<p_2<\cdots<\lim_{k\to\infty}p_k=\infty$.

We consider initial conditions of the form

$$\mathbf{x}(r)=\phi(r),\quad r\in\mathbb{Z}_0^- \tag{5}$$

with the sequence $\{\phi(r)\}_{r=-\infty}^{0}$ bounded with respect to the norm

$$\|\phi\|_\infty = \max_{i=\overline{1,n}}\left(\sup_{r\in\mathbb{Z}_0^-}|\phi_i(r)|\right).$$

The qualitative analysis of neural dynamics plays an important role in the design of practical neural networks. To solve problems of optimization, neural control and signal processing, neural networks have to be designed in such a way that, for a given external input, they exhibit only one globally asymptotically stable steady state. Referring to continuous-time neural networks with distributed delays and impulses, this matter has been treated in Huang, Luo, and Yang (2007), Huang, Wang, and Xia (2008), Kelin, Zuoan, and Qiankun (2007), Li (2009); Li, Hua, and Fei (2009), Li and Yang (2006), Li, Zhang, and Li (2009), Liu and Huang (2006), Mohamad, Gopalsamy, and Akça (2008), Ping and Lu (2009), Wang, Xiong, Zhou, Xiao, and Yu (2006), Xia, Huang, and Han (2008), Yin and Li (2009) and Zhou (2009); Zhou and Li (2009). As for discrete-time neural networks with impulses, we refer to (Akça, Alassar, Covachev, & Covacheva, 2004; Akça, Alassar, Covachev, & Yurtsever, 2007; Huo & Li, 2009; Song & Cao, 2008; Zhang & Chen, 2008; Zhao, 2009; Zhou, Li, & Wan, 2009) and the references therein.

On the other hand, if neural networks are used to analyze associative memories, the existence of several locally asymptotically stable steady states is required (i.e. multistability), as they store information and constitute distributed and parallel neural memory networks. Many research results on multistability of continuous-time neural networks have been reported in Cao, Feng, and Wang (2008), Cheng, Lin, and Shih (2006), Cheng, Lin, and Shih (2007), Huang and Cao (2008a), Huang and Cao (2008b), Kaslik and Balint (2006), Shih and Tseng (2008), Wang, Lu, and Chen (2009), Yi, Tan, and Lee (2003) and Zhang, Yi, Yu, and Heng (2009). Multistability analysis is essentially different from mono-stability analysis. In mono-stability analysis, the objective is to derive conditions that guarantee that each network contains only one equilibrium point, and all the trajectories of the network converge to it. Whereas in multistability analysis, the networks are allowed to