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Complex and chaotic dynamics in a discrete-time-delayed Hopfield neural network with ring architecture

Eva Kaslik *, Stefan Balint

Department of Mathematics and Computer Science, West University of Timişoara, Bd. V. Pârvan nr. 4, 300223, Timişoara, Romania

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ABSTRACT

This paper is devoted to the analysis of a discrete-time-delayed Hopfield-type neural network of p neurons with ring architecture. The stability domain of the null solution is found, the values of the characteristic parameter for which bifurcations occur at the origin are identified and the existence of Fold/Cusp, Neimark–Sacker and Flip bifurcations is proved. These bifurcations are analyzed by applying the center manifold theorem and the normal form theory. It is proved that resonant 1:3 and 1:4 bifurcations may also be present. It is shown that the dynamics in a neighborhood of the null solution become more and more complex as the characteristic parameter grows in magnitude and passes through the bifurcation values. A theoretical proof is given for the occurrence of Marotto's chaotic behavior, if the magnitudes of the interconnection coefficients are large enough and at least one of the activation functions has two simple real roots.

1. Introduction

Since the pioneering work of Hopfield (1984), the dynamics of continuous-time Hopfield neural networks have been thoroughly analyzed. However, discrete-time counterparts of continuous-type neural networks have only been in the spotlight since 2000, even though they are essential when implementing continuous-time neural networks for practical problems such as image processing, pattern recognition and computer simulation.

In recent years, the theory of discrete-time dynamic systems has assumed a greater importance as a well-deserved discipline. Many results in the theory of difference equations have been obtained as natural discrete analogs of corresponding results from the theory of differential equations. Nevertheless, the theory of difference equations is a lot richer than the corresponding theory of differential equations. For example, a simple difference equation resulting from a first order differential equation may exhibit chaotic behavior which can only happen for higher order differential equations. This is the reason why, when studying discrete-time counterparts of continuous neural networks (Mohamad & Gopalsamy, 2000), important differences and more complicated behavior may also be revealed.

The analysis of the dynamics of neural networks focuses on three directions: discovering equilibrium states and periodic or quasi-periodic solutions (of fundamental importance in biological and artificial systems, as they are associated with central pattern generators (Pasemann, Hild, & Zahedi, 2003)), establishing stability properties and bifurcations (leading to the discovery of periodic solutions), and identifying chaotic behavior (with valuable applications to practical problems such as optimization (Chen & Aihara, 1995, 1997, 2001; Chen & Shih, 2002), associative memory (Adachi & Aihara, 1997) and cryptography (Yu & Cao, 2006)).

We refer to Guo, Huang, and Wang (2004) and Guo and Huang (2004) for the study of the existence of periodic solutions of discrete-time Hopfield neural networks with delays and the investigation of exponential stability properties.

In Yuan, Hu, and Huang (2004, 2005) and in the most general case, in He and Cao (2007), a bifurcation analysis of two-dimensional discrete neural networks without delays has been undertaken. In Zhang and Zheng (2005, 2007), the bifurcation phenomena have been studied, for the case of two- and n-dimensional discrete neural network models with multi-delays obtained by applying the Euler method to a continuous-time Hopfield neural network with no self-connections. In Guo, Tang, and Huang (2008) and Kaslik and Balint (2007, 2008b, 2009), a bifurcation analysis for discrete-time Hopfield neural networks of two neurons with self-connections has been presented, in the case of a single delay, two, three and four delays.

The latest results concerning chaotic dynamics in classical discrete-time-delayed neural networks of two neurons have been reported by Huang and Zou (2005) and Kaslik and Balint (2008a).

Ring architectures have been found in a variety of neural structures, such as hippocampus, cerebellum, neocortex, and even in chemistry and electrical engineering. The real cortical

* Corresponding author.
E-mail address: kaslik@info.uvt.ro (E. Kaslik).

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connectivity pattern is extremely sparse: most connections are between nearby cells, and long-range connections become progressively more infrequent. Hence, the ring represents a simplified connectivity structure, and ring neural networks are studied to gain insight into the mechanisms underlying the behavior of recurrent networks (Hirsch, 1989).

Different aspects (stability, bifurcations, patterns of nonlinear oscillations, nonlinear waves, synchronization) of continuous-time dynamics of rings of neurons with delays have been studied by Baldi and Atiya (1994), Campbell, Ruan, and Wei (1999), Guo (2005), Guo and Huang (2003), Guo and Huang (2006), Guo and Huang (2007a, 2007b), Bungay and Campbell (2007), Campbell, Ncube, and Wu (2006), Lu and Guo (2008), Wei and Jiang (2006) and Wei and Zhang (2008). Most of these studies have concerned lower dimensional networks (of three or four neurons) and/or systems with a single time delay.

Recently, Xu (2008) obtained delay-dependent conditions for the global asymptotic stability of the equilibrium of a continuous-time bidirectional delayed ring neural network, using Lyapunov's method, as well as conditions for the global existence of periodic solutions. The same paper also presents a numerical investigation of two possible routes towards chaotic behavior (via period-doubling bifurcations and via bifurcations from the quasi-periodic solutions). However, there is no known result concerning a theoretical proof of chaotic behavior in continuous-time ring neural networks.

Studying discrete-time dynamics of neural rings is a challenging task, which, to the best of our knowledge, has not yet been explored. This is the reason why, in this paper, we extend the results obtained for two-dimensional neural networks (Kaslik & Balint, 2008a) to discrete-time-delayed Hopfield-type neural networks of $p \geq 2$ neurons with ring architecture, described by:

$$\begin{cases} x_1(n+1) = ax_1(n) + T_1 g_1(x_p(n-k_p)) \\ x_2(n+1) = ax_2(n) + T_2 g_2(x_1(n-k_1)) \\ \cdots \\ x_p(n+1) = ax_p(n) + T_p g_p(x_{p-1}(n-k_{p-1})) \end{cases} \quad \forall n \geq \max(k_1, k_2, \ldots, k_p). \tag{1}$$

In this system, $a \in (0, 1)$ is the internal decay of the neurons, T_i are the interconnection coefficients, $g_i : \mathbb{R} \to \mathbb{R}$ represent the neuron input–output activations and $k_i \geq 0$ represent the delays. The activation functions g_i are of class C^3 and $g_i(0) = 0$.

System (1) is the discrete-time counterpart of the continuous-time ring neural network studied by Wei and Zhang (2008).

Throughout the paper, we will denote $b_i = T_i g_i'(0)$, $b = b_1 b_2 \ldots b_p$ and $k = k_1 + k_2 + \cdots + k_p$. Moreover, the notation $j = \overline{a, b}$ is used to express that $j \in \{a, a+1, \ldots, b\}$ (where $a, b \in \mathbb{N}$, $a < b$).

The aim of this paper is to present a complete stability and bifurcation analysis of system (1) in a neighborhood of the null solution, and to theoretically prove that chaotic behavior occurs when the absolute values of the interconnection coefficients are large enough and at least one of the activation functions has two simple real roots.

2. Stability and bifurcation results

The characteristic equation of the linearized system of (1) at the origin (null solution) is obtained by searching for nontrivial solutions of the form $x_i(n) = c_i z^n$, $i = \overline{1, p}$ for the linearized system. This leads to the following characteristic equation:

$$z^k (z-a)^p = b. \tag{2}$$

We denote by D_S the domain of stability of the null solution of (1), i.e. the set of the values of parameter b for which the null solution is asymptotically stable.

Proposition 1. *$z = e^{i\theta}$, $\theta \in [0, \pi]$ is a root of the characteristic equation (2) if and only if one of the following hold:*

- $\theta = \theta_0 = 0$ *and* $b = b_0 = (1-a)^p$
- $\theta = \theta_j = h^{-1}(j\pi)$ *and* $b = b_j = (-1)^j (1 + a^2 - 2a\cos\theta_j)^{p/2}$ *for* $j = \overline{1, p+k-1}$
- $\theta = \theta_{p+k} = \pi$ *and* $b = (-1)^{p+k}(1+a)^p$

where $h : (0, \pi) \to (0, (p+k)\pi)$ is the increasing bijective function defined by $h(t) = kt + p\cot^{-1}\left(\frac{\cos t - a}{\sin t}\right)$, where $\cot^{-1}$ denotes the inverse of the cotangent function restricted to the interval $(0, \pi)$.

Proof. Suppose that $z = e^{i\theta}$ is a root of the characteristic equation (2). Passing to the imaginary part in Eq. (2) it can be easily seen that there exists $j \in \{0, 1, \ldots, p+k\}$ such that $\theta = \theta_j$. Passing to the real part in Eq. (2), we obtain $b = b_j$. □

Lemma 2. *The values b_j given by Proposition 1 satisfy the following relations:*

i. $|b_0| < |b_1| < |b_2| < \cdots < |b_{p+k}|$
ii. $\text{sign}(b_j) = (-1)^j$ *for any* $j = \overline{0, p+k}$
iii. *if k is even then* $b_{p+k-1} < b_{p+k-3} < \cdots < b_3 < b_1 < 0 < b_0 < b_2 < \cdots < b_{p+k-2} < b_{p+k}$
iv. *if k is odd then* $b_{p+k} < b_{p+k-2} < \cdots < b_3 < b_1 < 0 < b_0 < b_2 < \cdots < b_{p+k-3} < b_{p+k-1}$.

Proof. These properties follow directly from the fact that $b = b_j = (-1)^j (1 + a^2 - 2a\cos\theta_j)^{p/2}$ for $j = \overline{0, p+k}$. □

Lemma 3. *Let $j \in \{0, 1, \ldots, k+p\}$ and the root $z(b)$ of the characteristic equation (2) satisfying $z(b_j) = z^\star = e^{i\theta_j}$. Then $(-1)^j \frac{d|z|}{db}\big|_{b=b_j} > 0$.*

Proof. Indeed, denoting $P(z) = z^{k-1}(z-a)^{p-1}[(p+k)z - ak]$, one has:

$$\frac{d|z|^2}{db} = \bar{z}\frac{dz}{db} + z\frac{d\bar{z}}{db} = \frac{2\text{Re}(zP(z))}{|P(z)|^2}.$$

Therefore

$$\left.\frac{d|z|^2}{db}\right|_{b=b_j} = \frac{2\text{Re}(z^\star P(z^\star))}{|P(z^\star)|^2} = \frac{b_j}{|P(z^\star)|^2}\left(k + p\frac{1 - a\cos\theta_j}{1 + a^2 - 2a\cos\theta_j}\right).$$

Hence, $\text{sign}\left(\frac{d|z|}{db}\big|_{b=b_j}\right) = \text{sign}(b_j) = (-1)^j$, which completes the proof. □

Remark 4. According to Lemma 3, the number $\sigma(b)$ of the multipliers of system (1) outside the unit circle, regarded as a function of the parameter b, is increasing on the interval $[0, \infty)$ and decreasing on the interval $(-\infty, 0]$. One has:

$$\sigma(b) = \begin{cases} p+k, & b \in (-\infty, \min(b_{p+k-1}, b_{p+k})) \\ 2j, & b \in [b_{2j+1}, b_{2j-1}), j = \overline{1, \left[\frac{p+k-1}{2}\right]} \\ 0, & b \in [b_1, b_0] \\ 2j+1, & b \in (b_{2j}, b_{2j+2}], j = \overline{0, \left[\frac{p+k-2}{2}\right]} \\ p+k, & b \in (\max(b_{p+k-1}, b_{p+k}), \infty). \end{cases}$$

Proposition 5. *The null solution of (1) is asymptotically stable if and only if*

$$b \in D_S = (b_1, b_0).$$

At the boundary of the stability domain D_S, the following bifurcation