Knowledge-Based Systems 52 (2013) 194–200

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

Distributivity equations and Mayor's aggregation operators

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ARTICLE INFO

Article history:
Received 21 January 2013
Received in revised form 30 July 2013
Accepted 2 August 2013
Available online 16 August 2013

Keywords:
Aggregation operators
Uninorm
Nullnorm
Absorbing element
Neutral element
Distributivity equations

ABSTRACT

The focus of this paper are distributivity equations involving the binary aggregation operators on the unit interval [0,1] with either absorbing or neutral element from the open interval (0,1), and the Mayor's aggregation operators from [28]. In the second part of this paper, problem is extended to aggregation operators that have neither neutral nor absorbing element.

1. Introduction

Aggregation operators play an important role in many different theoretical and practical fields (fuzzy sets theory, theory of optimization, operations research, information theory, engineering design, game theory, voting theory, integration theory, etc.), particularly in decision making theory (see [2,15,17,21,23]). Lately, a high level of attention is directed towards characterizations of pairs of aggregation operators that are satisfying the distributivity law. Investigation of this problem has roots in [1] and, in recent years, it has been focused on t-norms and t-conorms [15], aggregation operators, quasi-arithmetic means [5], pseudo-arithmetical operations [3], fuzzy implications [30,31], uninorms and nullnorms [8,14,25,26,33]. Additionally, many authors are considering distributivity inequalities [9,10], as well as distributivity equations on a restricted domain [4,12,13,18,20–22,29,32]. An interesting application of this restricted setting on two Borel-Cantelli lemmas and independence of events for decomposable measures is given in [7].

The aim of this paper is to extend research from [5] towards binary aggregation operators that have either an absorbing element or a neutral element from $(0,1)$. In [5] the previous problem is solved for one special case, i.e., when neutral elements are limited to 1 and 0 (t-norms and t-conorms). Furthermore, the presented research also extends results form [5] towards non-commutative and non-associative operators. This line of research presents a contemporary topic (see [20,33]) that is highly interesting since it opens some new possibilities in the utility theory (see [19]). Therefore, the main concern of this paper is how to solve functional equations

$$F(x, G(y,z)) = G(F(x,y), F(x,z)), \quad x,y,z \in [0,1]$$

and

$$F(G(y,z), x) = G(F(y,x), F(z,x)), \quad x,y,z \in [0,1]$$

where one of unknown functions is an aggregation operator defined in the sense of G. Mayor (see [28]), and another one is either a relaxed uninorm or a relaxed nullnorm [8]. The second part of this paper contains even further extension of this problem involving aggregation operators that have neither neutral nor absorbing element. Also, results presented in this paper are additionally clarifying structure of the observed GM-operators. Since paper [5] has considered only min and max as options for the GM-operators, stricture of the GM-operators for other cases was not investigated, presented results present a step forward for this investigation.

This paper is organized as follows. Section 2 contains preliminary notions concerning aggregation operators defined in the sense of G. Mayor, aggregation operators with neutral and absorbing elements and distributivity equations. Results on distributivity between aggregation operator given in the sense of G. Mayor and relaxed nullnorm are given in the third section. Section 4 consists of results on distributivity between aggregation operator in the sense of G. Mayor and relaxed uninorm from the classes N_e^{max} and N_e^{min}. Topic of the fifth section is distributivity when one of the

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http://dx.doi.org/10.1016/j.knosys.2013.08.013

aggregation operators has neither neutral nor absorbing element. Some concluding remarks are given in the sixth section.

2. Preliminaries

A short overview of notions that are essential for this paper is given in this section [6,8,16,17,21,24,28,34].

2.1. Aggregation operators

First, let us recall the basic definition of an aggregation operator on $[0,1]$.

Definition 1 [17]. An aggregation operator is a function $A^{(n)}:[0,1]^n \to [0,1]$ that is nondecreasing in each variable and that fulfills the following boundary conditions

$$A^{(n)}(0,\dots,0) = 0 \quad \text{and} \quad A^{(n)}(1,\dots,1) = 1.$$

Of course, the previous definition of aggregation operators can be extended to an arbitrary real interval $[a,b]$. Perhaps the oldest example of an aggregation operator is the arithmetic mean defined by

$$A^{(n)}(x_1,\dots,x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

The integer n represents number of input values of the observed aggregation operator. Since the topic of this paper are the binary aggregation operators, they will be denoted simply by A instead of $A^{(2)}$. Many additional properties such as continuity, associativity, commutativity, idempotency, decomposability, autodistributivity, bisymmetry, and neutral and absorbing elements, etc., are often required for aggregation operators, depending on background in which the aggregation is performed (see [17]).

More accurately, the focus is on the binary aggregation operators introduced by Mayor in [28] that, for the sake of simplicity, will be refereed to as *the GM aggregation operators*, the nullnorms and the uninorms.

2.1.1. GM aggregation operators

Definition 2 [28]. A GM aggregation operator F is a commutative binary aggregation operator that satisfy the following boundary conditions for all $x \in [0,1]$:

$$F(0,x) = F(0,1)x \quad \text{and} \quad F(x,1) = (1 - F(0,1))x + F(0,1).$$

The following properties of the GM aggregation operators are essential for the further characterizations.

Theorem 3 [28]. *Let F be a GM aggregation operator. Then, the following holds:*

(i) F is associative if and only if F is a t-norm or t-conorm;
(ii) $F = \min$ or $F = \max$ if and only if $F(0,1) = 0$ or $F(0,1) = 1$ and $F(x,x) = x$ for all $x \in [0,1]$;
(iii) F is idempotent if and only if $\min \leqslant F \leqslant \max$.

2.1.2. Relaxed nullnorm

Another type of aggregation operators that will be used in this paper is the aggregation operator with an absorbing element, namely the relaxed nullnorm.

Definition 4 [6]. A nullnorm V is a binary aggregation operator on $[0,1]$ that is commutative, associative and for which there exists an element s in $[0,1]$ such that

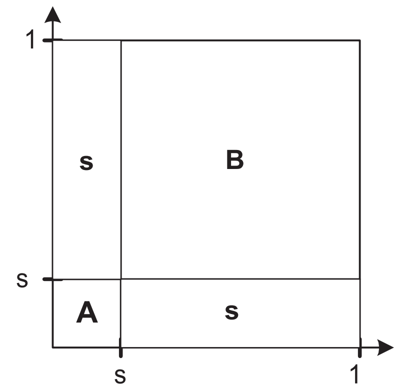

Fig. 1. An operator from Z_s.

$$V(x,0) = x \quad \text{for} \quad x \leqslant s \quad \text{and} \quad V(x,1) = x \quad \text{for} \quad x \geqslant s.$$

It is clear that s from the previous definition is an absorbing element: $V(x,s) = s$ for all x.

The previous definition, as special cases, contains definitions of triangular norms and triangular conorms. For $s = 0$ operator V is a t-norm denoted by T, and for $s = 1$ operator V is a t-conorm denoted by S.

The form of relaxed nullnorms that is obtained by omitting commutativity and associativity from the previous definition was introduced in [8]. Family of all such operators is denoted by Z_s. The following representation theorem for this type of aggregation operators with absorbing element (annihilator) was given in [8] (see Fig. 1).

Theorem 5 [8]. *Let $s \in [0,1]$. $G \in Z_s$ if and only if*

$$G = \begin{cases} A & \text{on}[0,s]^2, \\ B & \text{on}[s,1]^2, \\ s & \text{otherwise}, \end{cases} \tag{1}$$

where $A:[0,s]^2 \to [0,s]$ is a binary aggregation operator with neutral element 0 and $B:[s,1]^2 \to [s,1]$ is a binary aggregation operator with neutral element 1.

Remark 6. The only example of an idempotent operator from Z_s, $s \in (0,1)$, is obtained for $A = \max$ and $B = \min$. Obviously, operator of that form is a nullnorm, not only a relaxed nullnorm.

The following example presents a non-idempotent, non-associative and non-commutative operator from the class $Z_{\frac{1}{2}}$.

Example 7 [8]. Aggregation operator G given by the formula

$$G(x,y) = \begin{cases} x + y - 2xy & \text{for } (x,y) \in [\frac{1}{2}(1-y), \frac{1}{2}] \times [0, \frac{1}{2}) \\ \max(x,y) & \text{for } (x,y) \in [0, \frac{1}{2}(1-y)] \times [0, \frac{1}{2}) \\ \min(x,y) & \text{for } (x,y) \in (\frac{1}{2}, 1]^2 \\ \frac{1}{2} & \text{otherwise} \end{cases} \tag{2}$$

belongs to the class $Z_{\frac{1}{2}}$.

More general families of operators with absorbing element were studied in [27].

2.1.3. Relaxed uninorm

The following type of aggregation operators that are necessary for the presented research is an aggregation operator with a neutral element, namely the relaxed uninorm.