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Returns to scale and scale elasticity in the presence of weight restrictions and alternative solutions

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ABSTRACT

In this paper some new results about two important topics in performance analysis, including scale elasticity (SE) and returns to scale (RTS), in the presence of weight restrictions and alternative solutions are proved. Since SE and RTS help managers to make decisions about the expansion or contraction of the operation of decision making units under assessment, the established results can be useful from both theoretical and applied points of view. The provided implications are devoted to some mathematical characterizations and properties of SE and RTS as well as their relationships.

1. Introduction

For evaluating decision making units (DMUs), Charnes et al. [8] proposed the data envelopment analysis (DEA) technique, which allows any DMU to select their most favorable weights while requiring the resulting ratios of the sum of weighted outputs to the sum of weighted inputs of all DMUs to be less than or equal to a constant value. After introducing the first model in DEA, the CCR model by Charnes et al. [8], Banker et al. [4] developed the DEA technique by providing the BCC model. Nowadays DEA has allocated a wide variety of research in Operations Research (OR) to itself.

Two concepts that play a vital role in the theory of production are those of returns to scale (RTS) and scale elasticity (SE). RTS and SE can provide useful information on the optimal size of DMUs [12], or on whether small in size DMUs over- or under-perform larger ones, and vice versa, i.e., they are used to determine whether a technically efficient DMU can improve its productivity by resizing the scale of its operations.

One of the crucial topics in DEA literature is imposing the weight restrictions to models for incorporating the value judgments and opinions of the managers [1,16,23–25]. Although there are many papers in the DEA literature which discuss about the theory and applications of RTS and SE (see, e.g., [2–6,9–15,17–22,26,27] among others), studying these subjects in the presence of weight restrictions is very young. In only existing paper, Tone [24] provided a method for determining RTS in weight-restricted DEA models, after establishing some useful lemmas and theorems. In this paper some new mathematical characterizations of RTS and SE, in the presence of weight restrictions and alternative solutions and regarding the concept of multifunction, are provided. The provided results are useful from both theoretical and practical points of view and help us to have better applications of DEA. The rest of the paper unfolds as follows: Section 2 contains preliminaries; Section 3 gives the main results of the paper, and Section 4 contains some conclusions.

2. Preliminaries

Suppose that (x_j, y_j) for $j = 1, 2, \ldots, J$ are M-dimensional input and N-dimensional output vectors. Relative to the data set $\{(x_j, y_j) : j = 1, 2, \ldots, J\}$, we construct the $(N \times J)$ matrix of observed outputs, Y, and the $(M \times J)$ matrix of observed inputs, X. We assume that inputs and outputs are positive.

Following the existing literature we provide four sets on a J-dimensional vector of intensity variables λ:

$$LAM^V = \{\lambda : e\lambda = 1, \lambda \geqq 0\},$$
$$LAM^{NI} = \{\lambda : e\lambda \leqq 1, \lambda \geqq 0\},$$
$$LAM^{ND} = \{\lambda : e\lambda \geqq 1, \lambda \geqq 0\},$$
$$LAM^C = \{\lambda : \lambda \geqq 0\},$$

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where e is a row vector with all components equal to one. The above four sets are corresponding to variable, nonincreasing, nondecreasing, and constant RTS assumption of technology, respectively.

Now we recall some weight-restricted DEA models for evaluating a DMU under consideration, say DMU$_o$, where $o \in \{j = 1, 2, \dots, J\}$. An output-oriented V-WR model in multiplier form is constructed as follows:

$$v(x_o, y_o|\phi_W^V)x_o + w(x_o, y_o|\phi_W^V) = \min_{v,w,\mu}\{vx_o + w : -\mu Y + vX + we \geqq 0, \mu y_o = 1,\ vP \leqq 0,\ \mu Q \leqq 0,\ v \geqq 0,\ \mu \geqq 0, w \text{ free}\}, \tag{1}$$

where matrices P and Q are associated with weight restrictions. Here, we deal with homogeneous inequality constraints. This model called WR-BCC model or V-WR model which stands for a weight-restricted model under variable RTS (VRS) assumption of technology. The dual of (1), DWR-BCC model or V-DWR model, is:

$$\phi_W^V(x_o, y_o) = \max_{\phi,\lambda,\tau,\pi}\{\phi : Y\lambda + Q\tau - s^+ = \phi y_o, X\lambda - P\pi + s^- = x_o, \lambda \in LAM^V, \phi \text{ free}, \pi \geqq 0, \tau \geqq 0, s^- \geqq 0, s^+ \geqq 0\}, \tag{2}$$

where (x_o, y_o) is DMU$_o$'s input–output vector, and π and τ are dual variables corresponding to the third and fourth constraints of (1), respectively.

Also, replacing LAM^V by LAM^C, LAM^{NI}, and LAM^{ND} in (2), we obtain the C-DWR, NI-DWR, and ND-DWR models, respectively, as follows:

$$\phi_W^C(x_o, y_o) = \max_{\phi,\lambda,\tau,\pi}\{\phi : Y\lambda + Q\tau - s^+ = \phi y_o,\ X\lambda - P\pi + s^- = x_o, \lambda \in LAM^C, \phi \text{ free}, \pi \geqq 0, \tau \geqq 0, s^- \geqq 0, s^+ \geqq 0\}, \tag{3}$$

$$\phi_W^{NI}(x_o, y_o) = \max_{\phi,\lambda,\tau,\pi}\{\phi : Y\lambda + Q\tau - s^+ = \phi y_o, X\lambda - P\pi + s^- = x_o, \lambda \in LAM^{NI}, \phi \text{ free}, \pi \geqq 0, \tau \geqq 0, s^- \geqq 0, s^+ \geqq 0\}, \tag{4}$$

$$\phi_W^{ND}(x_o, y_o) = \max_{\phi,\lambda,\tau,\pi}\{\phi : Y\lambda + Q\tau - s^+ = \phi y_o, X\lambda - P\pi + s^- = x_o, \lambda \in LAM^{ND}, \phi\text{free}, \pi \geqq 0, \tau \geqq 0, s^- \geqq 0, s^+ \geqq 0\}. \tag{5}$$

Lemma 1.

(i) $1 \leqq \phi_W^V(x_o, y_o) \leqq \phi_W^{NI}(x_o, y_o) \leqq \phi_W^C(x_o, y_o)$.
(ii) $1 \leqq \phi_W^V(x_o, y_o) \leqq \phi_W^{ND}(x_o, y_o) \leqq \phi_W^C(x_o, y_o)$.
(iii) $\phi_W^V(x_o, y_o) = \min\{\phi_W^{NI}(x_o, y_o), \phi_W^{ND}(x_o, y_o)\}$.
(iv) $\phi_W^C(x_o, y_o) = \max\{\phi_W^{NI}(x_o, y_o), \phi_W^{ND}(x_o, y_o)\}$.

Definition 1 (*[9,24]*). DMU$_o(x_o, y_o)$ is V-WR-efficient, if and only if for all optimal solutions $(\phi_W^V, \lambda^V, \pi, \tau, s^-, s^+)$ of model (2) we have

$$\phi_W^V = 1, \quad s^- = 0, \quad s^+ = 0.$$

In all of the DEA interpretations the concept of the production possibility set (PPS) is important. Tone [24] has defined a PPS, P_W, under weight restrictions, as follows:

$$P_W = \{(x, y) : x \geqq X\lambda - P\pi, y \leqq Y\lambda + Q\tau, e\lambda = 1, \lambda \geqq 0, \pi \geqq 0, \tau \geqq 0\}. \tag{6}$$

In fact, P_W is under VRS assumption of technology and, regarding this, model (2) can be rewritten as:

$$\phi_W^V(x_o, y_o) = \max_{\phi,\lambda,\tau,\pi}\{\phi|(x_o, \phi y_o) \in P_W\}.$$

DEA categorizes DMUs into three classes according to their RTS classification: Constant RTS (CRS), Increasing RTS (IRS), and Decreasing RTS (DRS); and the RTS classification of DMUs can be used to improve the operation of the units. Note that, hereafter the notations NIRS and NDRS stand for nonincreasing RTS and nondecreasing RTS, respectively. Also, hereafter the notations cl, ∂ and int stand for closure, boundary and interior of the related sets, respectively. The following definition is an improved version of that provided in [4].

Definition 2. Assume that (x_o, y_o) is V-WR-efficient, then

(i) IRS prevails at (x_o, y_o), if there exists a $\delta^* > 0$ such that

$$((1 + \delta)x_o, (1 + \delta)y_o) \in int\ P_W$$

for each $\delta \in (0, \delta^*)$ and

$$((1 + \delta)x_o, (1 + \delta)y_o) \notin int\ P_W$$

for each $\delta \in (-\delta^*, 0)$.

(ii) DRS prevails at (x_o, y_o), if there exists a $\delta^* > 0$ such that

$$((1 + \delta)x_o, (1 + \delta)y_o) \in int\ P_W$$

for each $\delta \in (-\delta^*, 0)$ and

$$((1 + \delta)x_o, (1 + \delta)y_o) \notin int\ P_W$$

for each $\delta \in (0, \delta^*)$.

(iii) CRS prevails at (x_o, y_o), if there exists a $\delta^* > 0$ such that at least one of the following conditions holds

(iii-a)
$((1 + \delta)x_o, (1 + \delta)y_o) \in \partial P_W$ for each $\delta \in (-\delta^*, \delta^*)$,

(iii-b)
$((1 + \delta)x_o, (1 + \delta)y_o) \notin \partial P_W$ for each $\delta \in (-\delta^*, \delta^*) - \{0\}$,

(iii-c) $((1 + \delta)x_o, (1 + \delta)y_o) \in \partial P_W$

for each $\delta \in (0, \delta^*)$ and

$((1 + \delta)x_o, (1 + \delta)y_o) \notin \partial P_W$

for each $\delta \in (-\delta^*, 0)$,

(iii-d) $((1 + \delta)x_o, (1 + \delta)y_o) \notin \partial P_W$

for each $\delta \in (0, \delta^*)$ and

$((1 + \delta)x_o, (1 + \delta)y_o) \in \partial P_W$

for each $\delta \in (-\delta^*, 0)$.

Tone [24] provided the following theorem which is an extension of the results of [4] and relates the concept of returns to scale explained above closely to the sign of w in the optimal solutions of V-WR-model (1). Since w in the optimal solutions of model (1) is not unique, in the following theorem we use inf and sup of w under the optimal solutions set of (1) and denote these by $\underline{w}$ and $\bar{w}$, respectively.

Theorem 1. *Considering a V-WR-efficient DMU (x_o, y_o), we have*

(i) *DRS prevails at (x_o, y_o) if and only if $\underline{w} > 0$.*
(ii) *IRS prevails at (x_o, y_o) if and only if $\bar{w} < 0$.*
(iii) *CRS prevails at (x_o, y_o) if and only if $\underline{w} \leqq 0 \leqq \bar{w}$.*

Regarding Definition 2 and Theorem 1, we have the following theorem for V-WR-efficient units.

Theorem 2. *Considering (x_o, y_o) as a V-WR-efficient unit, we have:*

(i) DRS *prevails at (x_o, y_o) if $\phi_W^V(x_o, y_o) = \phi_W^{NI}(x_o, y_o) < \phi_W^C(x_o, y_o)$.*
(ii) IRS *prevails at (x_o, y_o) if $\phi_W^V(x_o, y_o) < \phi_W^{NI}(x_o, y_o) = \phi_W^C(x_o, y_o)$.*
(iii) CRS *prevails at (x_o, y_o) if $\phi_W^V(x_o, y_o) = \phi_W^{NI}(x_o, y_o) = \phi_W^C(x_o, y_o)$.*