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Global asymptotic stability analysis for delayed neural networks using a matrix-based quadratic convex approach

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ABSTRACT

This paper is concerned with global asymptotic stability for a class of generalized neural networks with interval time-varying delays by constructing a new Lyapunov–Krasovskii functional which includes some integral terms in the form of $\int_{t-h}^{t}(h-t-s)^j\dot{x}^T(s)R_j\dot{x}(s)ds$ $(j = 1, 2, 3)$. Some useful integral inequalities are established for the derivatives of those integral terms introduced in the Lyapunov–Krasovskii functional. A matrix-based quadratic convex approach is introduced to prove not only the negative definiteness of the derivative of the Lyapunov–Krasovskii functional, but also the positive definiteness of the Lyapunov–Krasovskii functional. Some novel stability criteria are formulated in two cases, respectively, where the time-varying delay is continuous uniformly bounded and where the time-varying delay is differentiable uniformly bounded with its time-derivative bounded by constant lower and upper bounds. These criteria are applicable to both static neural networks and local field neural networks. The effectiveness of the proposed method is demonstrated by two numerical examples.

1. Introduction

During the past decades, neural networks (NNs) have found a wide range of applications in a variety of areas such as associative memory (Bao, Wen, & Zeng, 2012; Michel, Farrell, & Sun, 1990; Zeng & Wang, 2010), static image processing (Chua & Yang, 1988), pattern recognition (Wang, 1995), and combinatorial optimization (Chen & Fang, 2000). It is true that most applications of NNs are closely dependent on some dynamic behaviors, especially on global asymptotic stability. However, due to the finite switching speeds of amplifiers, time delays are frequently encountered in practical NNs and they often degrade the system performance or destabilize an NN under consideration. Therefore, in recent years, increasing attention has been paid to stability of delayed NNs and a number of delay-dependent stability criteria have been reported in the literature, see for example, Faydasicok and Arik (2012, 2013), He, Wu, and She (2006), Shao (2008a), Wang and Chen (2012), Wang, Liu, and Liu (2009) and Zhang, Tang, Fang, and Wu (2012).

Consider the following delayed NN, whose equilibrium point is supposed to be shifted into the origin

$$\begin{cases}\dot{x}(t) = -Ax(t) + W_0 f(W_2x(t)) + W_1 f(W_2 x(t-\tau(t))) \\ x(\theta) = \phi(\theta), \quad \theta \in [-h_2, 0]\end{cases} \tag{1}$$

where $x(t) = \text{col}\{x_1(t), x_2(t), \dots, x_n(t)\} \in \mathbb{R}^n$ and $f(x(t)) = \text{col}\{f_1(x_1(t)), f_2(x_2(t)), \dots, f_n(x_n(t))\} \in \mathbb{R}^n$ are the neuron state vector and the neuron activation function, respectively; $A = \text{diag}\{a_1, a_2, \dots, a_n\} > 0$ is a constant real matrix; W_0, W_1 and W_2 are the interconnection matrices representing the weighting coefficients of the neurons; ϕ is an initial condition and the time-varying delay $\tau(t)$ is a continuous function satisfying

$$0 \le h_1 \le \tau(t) \le h_2 < \infty. \tag{2}$$

The NN model (1) includes some NNs as its special cases. If taking $W_2 = I$, then the model (1) represents a class of delayed local field neural networks (LFNNs) (Faydasicok & Arik, 2012; Liu, Wang, & Liu, 2009; Shao, 2008b; Zeng, He, Wu, & Zhang, 2011); If taking $W_0 = W_1 = I$, then the model (1) reduces to a class of delayed static neural networks (SNNs) (Li, Gao, & Yu, 2011; Zuo, Yang, & Wang, 2010). The study on delay-dependent stability of (1) aims to derive a maximum upper bound $h_2^{\max}$ of h_2 for a given $h_1 \ge 0$ such that the NN (1) is globally asymptotically stable for any $\tau(t)$ satisfying $h_1 \le \tau(t) \le h_2^{\max}$. The obtained $h_2^{\max}$ is thus regarded as a key

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index to measure the conservatism of a delay-dependent stability criterion and the larger $h_2^{\max}$, the less conservatism (Souza, 2013). In order to formulate some less conservative stability conditions, several effective approaches have been proposed in the past decade. To mention a few, one can refer to a free weighting matrix approach, a convex delay analysis approach, a delay-decomposition approach and a reciprocally convex approach. Recently, a new "quadratic convex approach" has been proposed in (Kim, 2011) to study the stability of linear systems with time-varying delays. This approach is then employed to investigate the global asymptotic stability of NNs in Zhang, Yang, Liu, and Zhang (2013). The key idea is to construct a novel Lyapunov–Krasovskii functional (LKF) with the following integral term

$$\mathscr{T}(t) := \int_{t-h_2}^{t} \sum_{j=1}^{3} (h_2 - t + s)^j \dot{x}^T(s) R_j \dot{x}(s) ds. \tag{3}$$

The conspicuous feature of $\mathscr{T}(t)$ is that the integrand is the sum of the quadratic terms $\dot{x}^T(s)R_i\dot{x}(s)$ multiplied by $h_2 - t + s$ with degree of i ($i = 1, 2, 3$). As a result, the time derivative of the chosen LKF can be bounded by a quadratic convex function with respect to $\tau(t)$. By employing the quadratic convex approach, some stability criteria are derived in Kim (2011) and Zhang et al. (2013). However, there are several issues to be addressed, which are given in the following.

- Taking the time derivative of $\mathscr{T}(t)$, we have

$$\begin{aligned}\dot{\mathscr{T}}(t) &= \dot{x}^T(t)(h_2R_1 + h_2^2R_2 + h_2^3R_3)\dot{x}(t) - \int_{t-h_2}^{t} \Phi_1(s)ds \\ &\quad - \int_{t-h_2}^{t} \big\{2(h_2 - t + s)\Phi_2(s) \\ &\quad + 3(h_2 - t + s)^2\Phi_3(s)\big\} ds \end{aligned}\tag{4}$$

  where $\Phi_i(s) = \dot{x}^T(s)R_i\dot{x}(s)$ ($i = 1, 2, 3$). The estimation made by Kim (2011) and Zhang et al. (2013) on the integral terms in (4) needs to be reconsidered. The main drawbacks lie in two aspects: one is that some useful terms, i.e. $-2\int_{t-\tau(t)}^{t}(h_2 - \tau(t))\Phi_2(s)ds$ and $-3\int_{t-\tau(t)}^{t}[(h_2 - \tau(t))^2 + 2(h_2 - \tau(t))(\tau(t) - t + s)]\Phi_3(s)ds$, are overly bounded by zero; and the other one is the use of the so-called basic inequality, which certainly leads to conservative results;
- The constraint of positive definiteness is imposed on the augmented Lyapunov matrix P (i.e. $P > 0$) in Kim (2011) and Zhang et al. (2013), while this constraint is not necessary for the positive definiteness of the chosen LKF;
- The use of the quadratic convex approach is questionable. For instance, in Kim (2011), the quadratic convex approach is applied to a function $\xi_t^T[\Psi_0 + d(t)\Psi_1 + \Upsilon_d]\xi_t$ (see the proof of Theorem 1 in Kim (2011)), while this function may be not a quadratic function on the scalar $d(t)$ because ξ_t is a vector-valued function *implicitly* dependent on $d(t)$. The same case also happens in Zhang et al. (2013);
- When the lower bound h_1 of $\tau(t)$ is strictly greater than zero, the conditions obtained in Kim (2011) and Zhang et al. (2013) fail to make any conclusion on the stability of the system under consideration.

Therefore, based on the observation above, it is significant to establish some new integral inequalities for the integral terms in (4) and develop the quadratic convex approach to formulate some less conservative stability criteria, which motivates the current study.

In this paper, we will present a matrix-based quadratic convex approach to stability of a class of generalized NNs described by (1). First, some novel integral inequalities for the integral terms in (4) are established, where the over-bounding performed in Kim (2011) and Zhang et al. (2013) is no longer involved. Second, a matrix-based quadratic convex approach is applied to derive a sufficient condition such that the positive definiteness of the chosen LKF can be ensured. As a result, the constraint $P > 0$ in both Kim (2011) and Zhang et al. (2013) is removed. Third, the matrix-based quadratic convex approach is employed to formulate some less conservative stability criteria for NN (1) for two cases, respectively, where the time-varying delay $\tau(t)$ satisfies (2) and where $\tau(t)$ satisfies both (2) and $\mu_1 \le \dot{\tau}(t) \le \mu_2$ with μ_1 and μ_2 being two constants. Moreover, these stability criteria are applicable not only to LFNNs but also to SNNs. Finally, two numerical examples are given to demonstrate the effectiveness of the proposed results.

Throughout this paper, the notations are standard. The superscript 'T' stands for the transpose of a vector or a matrix. For an invertible matrix M, its inverse matrix is denoted by M^{-1}. For a real symmetric matrix P, $P > 0$ ($P \ge 0$) means that the matrix P is positive definite (positive semi-definite), and $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ represent the maximum and minimum eigenvalues of the matrix P, respectively. I and 0 mean an identity matrix and a zero matrix of appropriate dimensions, respectively. diag$\{\cdots\}$ and col$\{\cdots\}$ denote a block-diagonal matrix and a block-column vector, respectively.

2. Some novel integral inequalities and a matrix-based quadratic convex approach

In this section, we first establish some novel integral inequalities, and then introduce a matrix-based quadratic convex approach to delay-dependent stability analysis for delayed NNs.

2.1. Some novel integral inequalities

To begin with, we introduce the following lemmas.

Lemma 1. *Let α and β be real column vectors with dimensions of n_1 and n_2, respectively. For given real positive symmetric matrices $M_1 \in \mathbb{R}^{n_1 \times n_1}$ and $M_2 \in \mathbb{R}^{n_2 \times n_2}$, if $\begin{bmatrix} M_1 & S \\ S^T & M_2 \end{bmatrix} \ge 0$, then the following inequality holds for any scalar $\kappa > 0$ and matrix $S \in \mathbb{R}^{n_1 \times n_2}$*

$$-2\alpha^T S\beta \le \kappa\alpha^T M_1\alpha + \kappa^{-1}\beta^T M_2\beta. \tag{5}$$

Proof. The proof can be completed by noticing that

$$\begin{bmatrix} M_1 & S \\ S^T & M_2 \end{bmatrix} \ge 0 \iff \begin{bmatrix} \kappa M_1 & S \\ S^T & \kappa^{-1}M_2 \end{bmatrix} \ge 0.$$

Lemma 2 (Seuret & Gouaisbaut, 2013)**.** *For a given matrix $R > 0$, the following inequality holds for any continuously differentiable function $\omega : [a, b] \to \mathbb{R}^n$*

$$\int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du \ge \frac{1}{b-a}(\Gamma_1^T R\Gamma_1 + 3\Gamma_2^T R\Gamma_2) \tag{6}$$

where

$$\Gamma_1 := \omega(b) - \omega(a)$$

$$\Gamma_2 := \omega(b) + \omega(a) - \frac{2}{b-a}\int_a^b \omega(u)du.$$

It is clear to see that the inequality (6) provides a tighter lower bound for $\int_a^b \dot{\omega}^T(u)R\dot{\omega}(u)du$ than Jensen's inequality because $3\Gamma_2^T R\Gamma_2 > 0$ for $\Gamma_2 \ne 0$. Thus, the inequality (6) is an improvement over Jensen's inequality.