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New criterion of asymptotic stability for delay systems with time-varying structures and delays[☆]

Bo Liu [a,b], Wenlian Lu [c,d,e], Tianping Chen [f,*]

[a] Key Laboratory of Intelligent Perception and Image Understanding of Ministry of Education of China, Xidian University, Xi'an 710071, PR China
[b] Institute of Industrial Science, The University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, Japan
[c] Centre for Computational Systems Biology, Fudan University, Shanghai, PR China
[d] Laboratory of Mathematics for Nonlinear Science, School of Mathematical Sciences, Fudan University, Shanghai, PR China
[e] Department of Computer Science, The University of Warwick, Coventry CV4 7AL, United Kingdom
[f] The School of Computer/Mathematical Sciences, Fudan University, 200433, Shanghai, PR China

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ABSTRACT

In this paper, we study asymptotic stability of the zero solution of a class of differential systems governed by a scalar differential inequality with time-varying structures and delays. We establish a new generalized Halanay inequality for the asymptotic stability of the zero solution for such systems under more relaxed conditions than the existing ones. We also apply the theoretical results to the analysis of self synchronization in networks of delayed differential systems and obtain a more general sufficient condition for self synchronization.

1. Introduction

The applications of delay differential equations can be found in many areas including control systems, neural networks (Chen, Lu, & Chen, 2005; Shen & Wang, 2009), and many others. And a fundamental problem in these applications is to determine the stability of the solutions, which has been analyzed for decades. For example, in order to analyze the asymptotic stability of the zero solution of the following delay-differential equations with fixed delay $\tau > 0$,

$$\dot{x}(t) = -ax(t) + bx(t-\tau). \tag{1}$$

Halanay (1966) proved the following inequality which was later called *Halanay inequality*.

Proposition 1. *Let $x(t) > 0$, $t \in \mathbb{R}$, be a differentiable scalar function of t that satisfies*

$$\dot{x}(t) \le -ax(t) + b \sup_{t-\tau \le s \le t} x(s), \quad t \ge t_0 \tag{2}$$

$$x(t) = \psi(t), \quad t \le t_0 \tag{3}$$

with $a > b > 0$ being constants and $\psi(t) \ge 0$ continuous and bounded for $t \le t_0$, then there exist $k > 0$ and $\gamma > 0$ such that $x(t) \le ke^{-\gamma(t-t_0)}$. Hence $x(t) \to 0$ as $t \to \infty$.

Later on, this inequality has been extended to more general types of delay differential equations. For example, in Baker and Tang (1996), Wen, Yu, and Wang (2008), it has been proved.

Proposition 2. *Let $x(t) > 0$, $t \in \mathbb{R}$ be a differentiable scalar function that satisfies*

$$\dot{x}(t) \le -a(t)x(t) + b(t) \sup_{q(t) \le s \le t} x(s), \quad t \ge t_0, \tag{4}$$

$$x(t) = \psi(t), \quad t \le t_0, \tag{5}$$

where $\psi(t) > 0$ is bounded and continuous for $t \le t_0$, $a(t), b(t) \ge 0$ for $t \ge t_0$, $0 < q(t) \le t$ and $q(t) \to \infty$ as $t \to \infty$. If there exists $\sigma > 0$ such that

$$-a(t) + b(t) \le -\sigma < 0, \quad t \ge t_0, \tag{6}$$

then (i) $x(t) \le \sup_{-\infty < s \le t_0} |\psi(s)|$, (ii) $x(t) \to 0$ as $t \to \infty$.

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* Corresponding author. Tel.: +86 21 65642903; fax: +86 21 65646073.
E-mail addresses: liu7bo9@gmail.com (B. Liu), wenlian@fudan.edu.cn (W. Lu), tchen@fudan.edu.cn (T. Chen).

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In Mohamad and Gopalsamy (2000), the authors consider continuous and discrete time Halanay-type inequalities and further generalize the results of Baker and Tang (1996) to the case of distributed delays.

Proposition 3 (*Theorem 2.2 of Mohamad and Gopalsamy (2000)*). *Let $x(t)$, $t \in \mathbb{R}$ be a nonnegative function that satisfies*

$$\dot{x}(t) \le -a(t)x(t) + b(t)\int_0^\infty K(s)x(t-s)ds, \quad t > t_0,$$
$$x(t) = |\varphi(t)| \quad t \le t_0,$$

where $\varphi(s)$, $s \in (-\infty, t_0]$, $a(t)$ and $b(t)$, $t \in \mathbb{R}$ are nonnegative continuous and bounded functions; the delay kernel $K(\cdot) : [0,\infty) \to [0,\infty)$ satisfies

$$\int_0^\infty K(s)e^{\alpha s}ds < \infty,$$

for some positive number α. Suppose further that

$$a(t) - b(t)\int_0^\infty K(s)ds \ge \sigma, \quad t \in \mathbb{R}, \tag{7}$$

where $\sigma = \inf_{t\in\mathbb{R}}[a(t) - b(t)\int_0^\infty K(s)ds] > 0$. Then there exists a positive number $\tilde{\alpha}$ such that

$$x(t) \le \left(\sup_{s\le t_0} x(s)\right)e^{-\tilde{\alpha}(t-t_0)}, \quad t > t_0.$$

In Baker (2010), the author made some refinement on the decay rate of their previous works.

Generalized Halanay inequalities have also been developed in the stability analysis of delay differential systems. For example, in Chen (2001), Lu and Chen (2004), the authors proposed some variants of the Halanay inequality to solve the global stability of delayed Hopfield neural networks.

Particularly, in Chen and Lu (2003), Lu and Chen (2004), the following periodic and almost periodic integro-differential systems

$$\begin{aligned}\frac{du_i(t)}{dt} &= -d_i(t)u_i(t) + \sum_{j=1}^n a_{ij}(t)g_j(u_j(t)) \\ &\quad + \sum_{j=1}^n \int_0^\infty f_j(u_j(t-\tau_{ij}(t)-s))d_sK_{ij}(t,s) + I_i(t), \\ &\quad i = 1,2,\ldots,n,\end{aligned} \tag{8}$$

where $d_sK_{ij}(t,s)$ are Lebesgue–Stieltjes measures for each t, are discussed.

As a direct consequence of the main theorem in Lu and Chen (2004), we have

Proposition 4. *Suppose that $|g_j(x)| \le G_j|x|$ and $|f_j(x)| \le F_j|x|$. If there exist positive constants $\xi_1, \xi_2, \ldots, \xi_n, \alpha$ such that for all $t > 0$ and $i = 1,2,\ldots,n$,*

$$\begin{aligned}&-\xi_i(d_i(t)-\alpha) + \sum_{j=1}^n \xi_j G_j|a_{ij}(t)| \\ &+ \sum_{j=1}^n \xi_j F_j e^{\alpha\tau_{ij}(t)}\int_0^\infty e^{\alpha s}|d_sK_{ij}(t,s)| \le 0,\end{aligned} \tag{9}$$

then for any solution $u(t) = [u_1(t),\ldots,u_n(t)]$, $t > 0$ of the system (8) with $I_i(t) = 0$, $i = 1,\ldots,n$, we have

$$\max_{i=1,\ldots,n}|u_i(t)| \le \max_{i=1,\ldots,n}\max_{-\tau\le s\le 0}(e^{\alpha s}|u_i(s)|)e^{-\alpha t}. \tag{10}$$

In particular, when $n = 1$, $d_sK_{11}(t,s) = b(t)\delta(s)$, $\tau_{11}(t) = \tau(t)$, we have

Proposition 5 (*Also See Chen (2001)*). *Suppose $-(a(t)-\alpha) + |b(t)|e^{\alpha\tau(t)} \le 0$, then for any continuous scalar function $x(t) \ge 0$ that satisfies*

$$\begin{cases}\dot{x}(t) \le -a(t)x(t) + |b(t)|\sup_{s\ge 0} x(t-s), & t > 0,\\ x(t) = |\varphi(t)|, & t \le 0,\end{cases} \tag{11}$$

we have

$$|x(t)| \le \max_{-\tau\le s\le 0}(e^{\alpha s}|\phi(s)|)e^{-\alpha t}. \tag{12}$$

Instead, when $n = 1$, $d_sK_{11}(t,s) = b(t)k(s)ds$, $\tau_{11}(t) = 0$, we have

Proposition 6. *Suppose $-(a(t)-\alpha) + b(t)\int_0^\infty e^{\alpha s}K(s)ds \le 0$, then for any continuous scalar function $x(t)$ satisfying*

$$\begin{cases}\dot{x}(t) \le -a(t)x(t) + b(t)\int_0^\infty K(s)x(t-s)ds, & t > 0,\\ x(s) = |\varphi(s)|, & t \le 0,\end{cases} \tag{13}$$

we have

$$|x(t)| \le \max_{-\tau\le s\le 0}(e^{\alpha s}|\phi(s)|)e^{-\alpha t}. \tag{14}$$

For more recent works, refer to Gil' (2013) and Liu, Lu, and Chen (2011a), Liu, Lu, and Chen (2012). In all the above mentioned works, there is a basic requirement: $\underline{a(t) > b(t)}$ for all t. This requirement is not satisfied in many real systems. For example, it is well known that a system switching among several subsystems can be stable even if not all the subsystems are stable. So it is necessary, if possible, to further generalize the Halanay inequality so that it can be used for more general cases.

In this paper, we will first generalize the differential inequalities with bounded time-varying delays under more relaxed requirements, say, without $a(t) > b(t)$ for all t. Then, we provide two applications of the theoretical results. First, we apply the theoretical results to the analysis of self synchronization in neural networks. Based on our new generalized Halanay inequality, we proved new sufficient conditions for self synchronization in neural networks with bounded time-varying delays. Then, we investigate periodic solutions of neural networks with periodic coefficients and time delays. Under more relaxed requirement, we proved new sufficient conditions for the existence and exponential stability of the periodic solutions of such neural networks.

The rest of the paper is organized as follows. In Section 2, the new generalized Halanay inequality is proposed and proved; two applications of the theoretical results are given in Section 3; numerical examples with simulations are given in Section 4; the paper is concluded in Section 5.

2. Generalized Halanay inequality

Consider a scalar function $x(t)$ governed by the inequality

$$\begin{cases}D^+|x(t)| \le -a(t)|x(t)| + b(t)\sup_{t-\tau_{\max}\le s\le t}|x(t-s)|, & t \ge 0,\\ x(s) = \phi(s), & s \in [-\tau_{\max}, 0],\end{cases} \tag{15}$$

where D^+ represents the upper right Dini derivative, $a(\cdot)$: $\mathbb{R}^+ \mapsto \mathbb{R}^+$, $b(\cdot)$: $\mathbb{R}^+ \mapsto \mathbb{R}$ are piecewise continuous and uniformly bounded, i.e., there exists $M_a > 0$, $M_b > 0$ such that $0 < a(t) \le M_a$, $|b(t)| \le M_b$, $\phi(s) \ge 0$ is the initial value, and $\tau(\cdot)$: