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Global exponential stability of neural networks with non-smooth and impact activations

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ABSTRACT

In this paper, we consider a model of impulsive recurrent neural networks with piecewise constant argument. The dynamics are presented by differential equations with discontinuities such as impulses at fixed moments and piecewise constant argument of generalized type. Sufficient conditions ensuring the existence, uniqueness and global exponential stability of the equilibrium point are obtained. By employing Green's function we derive new result of existence of the periodic solution. The global exponential stability of the solution is investigated. Examples with numerical simulations are given to validate the theoretical results.

1. Introduction

Recurrent neural networks and impulsive recurrent neural networks have been investigated due to their extensive applications in classification of patterns, associative memories, image processing, optimization problems, and other areas (Chua & Roska, 1992; Chua & Yang, 1988a, 1988b; Civalleri, Gilli, & Pandolfi, 1993; Coombes & Laing, 2009; Gopalsamy, 2004; Guan & Chen, 1999; Guan, Lam, & Chen, 2000; Hopfield, 1984; Michel, Farrell, & Porod, 1989; Mohammad, 2007; Xu & Yang, 2005; Yang & Cao, 2007; Zhang & Sun, 2005). It is well known that these applications depend crucially on the dynamical behavior of the networks. For example, if a neural network is employed to solve some optimization problems, it is highly desirable for the neural network to have a unique globally stable equilibrium (Cao, 1999; Cao & Zhou, 1998; Driessche & Zou, 1998; Huang, Cao, & Wang, 2002; Li, Huang, & Zhu, 2003; Xu, Chu, & Lu, 2006; Yucel & Arik, 2004; Zeng & Wang, 2006). Therefore, stability analysis of neural networks has received much attention and various stability conditions have been obtained over the past years. Another interesting subject is to study the dynamical behavior of existence of the periodic solutions in recurrent neural networks. These periodic solutions present periodic pattern and have been used in learning theory, which are meant to capture the idea that certain activities or motions are learned by repetition (Huang, Hob, & Cao, 2005; Townley et al., 2000).

In numerical simulations and practical implementations of neural networks, it is essential to formulate a discrete-time system, an analogue of the continuous-time system. Hence, stability for discrete-time neural networks has also received considerable attention from many researchers (Barabanov & Prokhorov, 2002; Liang, Cao, & Lam, 2005; Liu, Wang, & Liu, 2009; Liu, Wang, Serrano, & Liu, 2007; Mohamad, 2001; Zhao, 2009). As we know, the reduction of differential equations with piecewise constant argument to discrete equations has been the main and possibly a unique way of stability analysis for these equations (Cooke & Wiener, 1984; Wiener, 1993). Hence, one has not investigated the problem of stability completely, as only elements of a countable set were allowed to be discussed for initial moments. Finally, only equations which are linear with respect to the values of solutions at non-deviated moments of time have been investigated. That narrowed significantly the class of systems. In papers (Akhmet, 2006, 2007, 2008; Akhmet & Aruğaslan, 2009), the theory of differential equations with piecewise constant argument has been generalized by Akhmet. Later, Akhmet gathered all results for this type of differential equations in the book (Akhmet, 2011). All of these equations are reduced to equivalent integral equations such that one can investigate many problems, which have not been solved properly by using discrete equations, i.e., existence and uniqueness of solutions, stability and existence of periodic

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solutions. Moreover, since we do not need additional assumptions on the reduced discrete equations, the new method requires more easily verifiable conditions, similar to those for ordinary differential equations.

In this paper, we develop the model of recurrent neural networks to differential equations with both impulses and piecewise constant argument of generalized type. It is well known that impulsive differential equation (Akhmet, 2010; Lakshmikantham, Bainov, & Simeonov, 1989; Samoilenko & Perestyuk, 1995) is one of the basic instruments so the role of discontinuity has been understood better for the real world problems. In real world, many evolutionary processes are characterized by abrupt changes at certain time. These changes are called to be impulsive phenomena, which are included in many fields such as biology involving thresholds, bursting rhythm models, physics, chemistry, population dynamics, models in economics, optimal control, etc. In the literature, recurrent neural networks have been developed by implementing impulses and piecewise constant argument (Akhmet, Aruğaslan, & Yılmaz, 2010a; Akhmet & Yılmaz, 2010; Gopalsamy, 2004; Guan & Chen, 1999; Guan et al., 2000; Mohammad, 2007; Xu & Yang, 2005; Yang & Cao, 2007; Zhang & Sun, 2005) issuing from different reasons: In implementation of electronic networks, the state of the networks is subject to instantaneous perturbations and experiences abrupt change at certain instants which may be caused by switching phenomenon, frequency change or other sudden noise. Furthermore, the dynamics of quasi-active dendrites with active spines is described by a system of point hot-spots (with an integrate-and-fire process), see Coombes and Bressloff (2000) and Timofeeva (2010) for more details. This leads to the model of recurrent neural network with impulses. It is important to say that the neighbor moments of impulses may depend on each other. For example, the successive impulse moment may depend upon its predecessor. The reason for this phenomenon is the interior design of a neural network. On the other hand, due to the finite switching speed of amplifiers and transmission of signals in electronic networks or finite speed for signal propagation in neural networks, time delays exist (Chua & Roska, 1992; Chua & Yang, 1988a; Civalleri et al., 1993; Coombes & Laing, 2009). Moreover, the idea of involving delayed arguments in the recurrent neural networks can be explained by the fact that we assume neural networks may "*memorize*" values of the phase variable at certain moments of time to utilize the values during middle process till the next moment. Thus, we arrive to differential equations with piecewise constant argument. Obviously, the distances between the "*memorized*" moments may be very variative. Consequently, the concept of a generalized type of piecewise constant argument is fruitful for recurrent neural networks (Akhmet et al., 2010a; Akhmet, Aruğaslan, & Yılmaz, 2010b; Akhmet & Yılmaz, 2010). Therefore, it is possible to apply differential equations with both impulses and piecewise constant argument to neural networks theory.

The intrinsic idea of the paper is that our model is not only from the applications point of view, but also from a new system of differential equations. That is, we develop differential equations with piecewise constant argument of generalized type to a new class of systems; impulsive differential equations with piecewise constant argument and apply them to recurrent neural networks (Akhmet, 2006, 2007, 2008; Akhmet & Aruğaslan, 2009; Akhmet et al., 2010a; Akhmet & Yılmaz, 2010). The main novelty of the paper is that the impact activators are considered with piecewise constant argument. Never before, they were not combined together in the same model such as recurrent neural networks; however, applications of impulses and piecewise constant argument were separately developed in Akhmet et al. (2010a, 2010b), Akhmet and Yılmaz (2010), Barabanov and Prokhorov (2002), Gopalsamy (2004), Guan and Chen (1999), Guan et al. (2000), Liang et al. (2005), Liu et al. (2009), Liu et al. (2007), Mohamad (2001), Mohammad (2007), Xu and Yang (2005), Yang and Cao (2007), Zhang and Sun (2005) and Zhao (2009). Another novelty of this paper is that the sequence of moments $\theta_k,\ k \in \mathbb{N}$, where the constancy of the argument changes, and the sequence of impulsive moments, τ_k, are different. More precisely, each moment $\tau_i,\ i \in \mathbb{N}$, is an interior point of an interval (θ_k, θ_{k+1}). This makes our results more powerful in applications sense and simultaneously more deep in theoretical sense. One should recognize that this intermittency causes technical difficulties which have not been observed when impulsive activators and those with piecewise constant argument were involved in neural networks models, separately.

2. Model description and preliminaries

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{R}^+ = [0, \infty)$ be the sets of natural and nonnegative real numbers, respectively, and denote a norm on $\mathbb{R}^m$ by $\|\cdot\|$ where $\|u\| = \sum_{i=1}^m |u_i|$. Fix two real valued sequences $\theta = \{\theta_k\},\ \tau = \{\tau_k\},\ k \in \mathbb{N},\ \tau \cap \theta = \phi$ such that $\theta_k < \theta_{k+1}$ with $\theta_k \to \infty$ as $k \to \infty$ and $\tau_k < \tau_{k+1}$ with $\tau_k \to \infty$ as $k \to \infty$, and there exist two positive numbers $\overline{\theta}, \underline{\tau}$ such that $\theta_{k+1} - \theta_k \le \overline{\theta}$ and $\underline{\tau} \le \tau_{k+1} - \tau_k,\ k \in \mathbb{N}$. The condition of the empty intersection is caused by the investigation reasons. Otherwise, the proof of auxiliary results needs several additional assumptions.

The main subjects under investigation in this paper are the following impulsive recurrent neural networks with piecewise constant argument of generalized type:

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^m b_{ij} f_j(x_j(t)) \\ &\quad + \sum_{j=1}^m c_{ij} g_j(x_j(\beta(t))) + d_i, \quad t \ne \tau_k \end{aligned} \tag{2.1a}$$

$$\Delta x_i |_{t=\tau_k} = I_{ik}(x_i(\tau_k^-)), \quad a_i > 0,\ i = 1, 2, \ldots, m,\ k \in \mathbb{N}, \tag{2.1b}$$

where $\beta(t) = \theta_k$ (see Fig. 1) if $\theta_k \le t < \theta_{k+1},\ k \in \mathbb{N},\ t \in \mathbb{R}^+$, is an identification function, $\Delta x_i(\tau_k)$ denotes $x_i(\tau_k) - x_i(\tau_k^-)$, where $x_i(\tau_k^-) = \lim_{h \to 0^-} x_i(\tau_k + h)$. Moreover, m corresponds to the number of units in a neural network, $x_i(t)$ stands for the state vector of the ith unit at time t, $f_j(x_j(t))$ and $g_j(x_j(\beta(t)))$ denote, respectively, the measures of activation to its incoming potentials of the unit j at time t and $\beta(t)$, b_{ij}, c_{ij}, d_i are real constants, b_{ij} means the strength of the jth unit on the ith unit at time t, c_{ij} infers the strength of the jth unit on the ith unit at time $\beta(t)$, d_i signifies the external bias on the ith unit and a_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when it is disconnected from the network and external inputs.

In the theory of differential equations with piecewise constant argument (Akhmet, 2006, 2007, 2008), we take the function $\beta(t) = \theta_k$ if $\theta_k \le t < \theta_{k+1}$, that is, $\beta(t)$ is right continuous. However, as it is usually done in the theory of impulsive differential equations, at the points of discontinuity τ_k of the solution, solutions are left continuous. Thus, the right continuity is more convenient assumption if one considers equations with piecewise constant arguments, and we shall assume the continuity for both, impulsive moments and moments of the switching of constancy of the argument.

We say that the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^m$ is from the set $PC_\tau(\mathbb{R}^+, \mathbb{R}^m)$ if:

(i) φ is right continuous on $\mathbb{R}^+$;
(ii) it is continuous everywhere except possibly moments τ where it has discontinuities of the first kind.

Moreover, we introduce a set of functions $PC_{\tau \cup \theta}(\mathbb{R}^+, \mathbb{R}^m)$ if we replace τ by $\tau \cup \theta$ in the last definition. In our paper, we understand